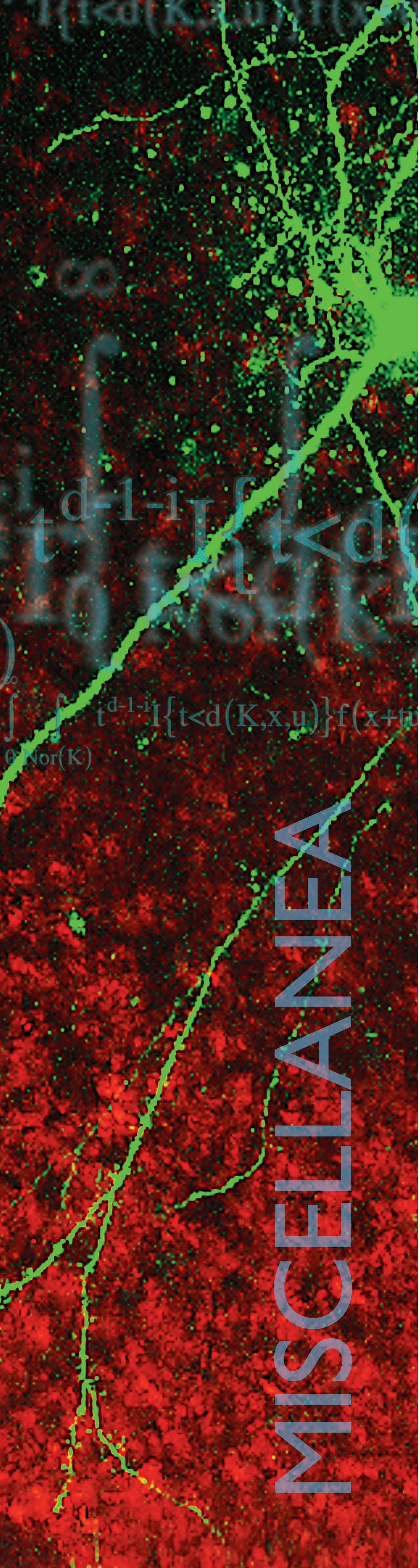


CENTRE FOR **STOCHASTIC GEOMETRY**
AND ADVANCED **BIOIMAGING**

Adrian Baddeley

Joining the dots
– inaugural lecture 7 November 2013

Miscellanea No. 08, November 2013

Joining the Dots

Statistics for spatial point patterns

Adrian Baddeley

Le Petit Journal

ADMINISTRATION

5 CENT. SUPPLÉMENT ILLUSTRÉ 5 CENT.

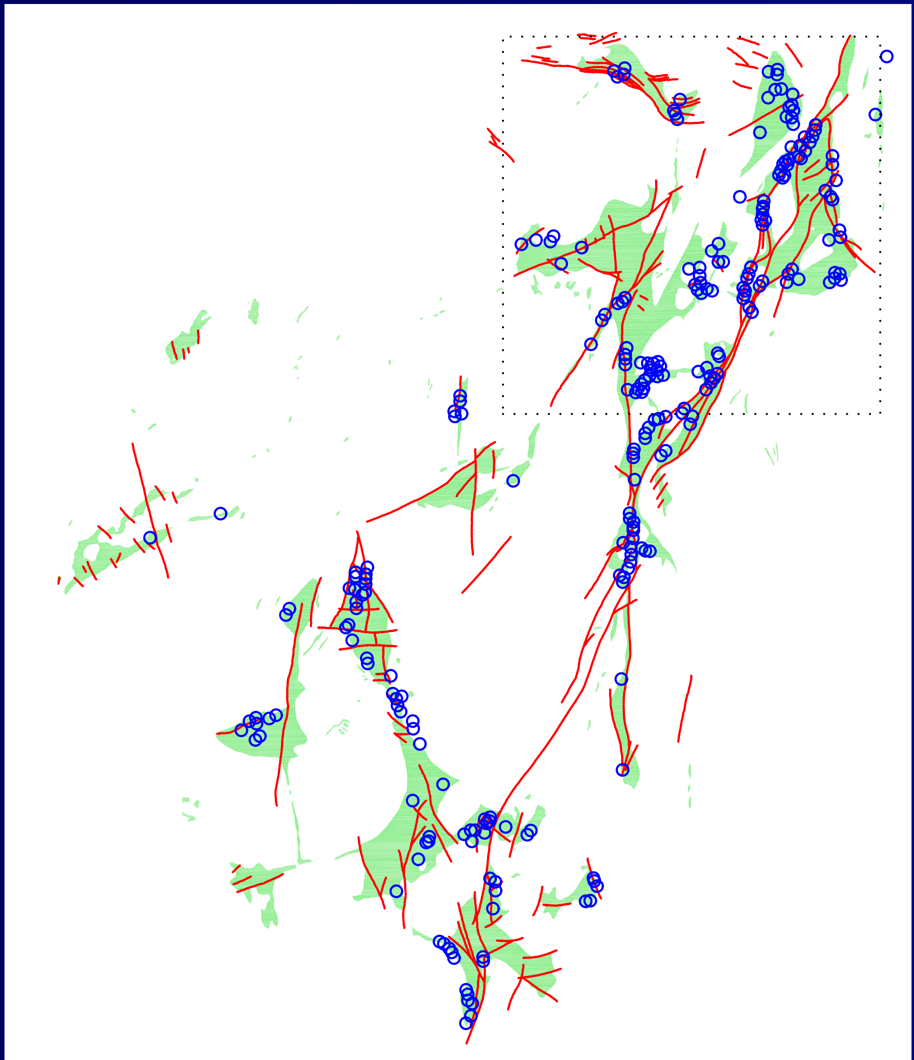
ABONNEMENTS

LE CHOLÉRA

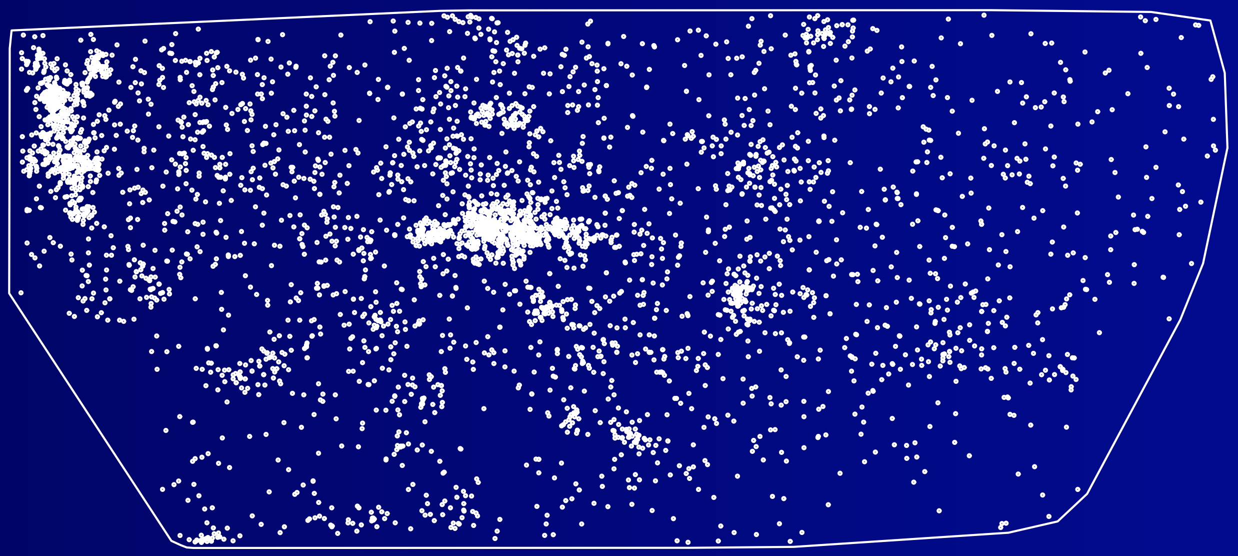
John Snow’s map of cholera in Soho, London, 1854

Cholera in Soho 1854

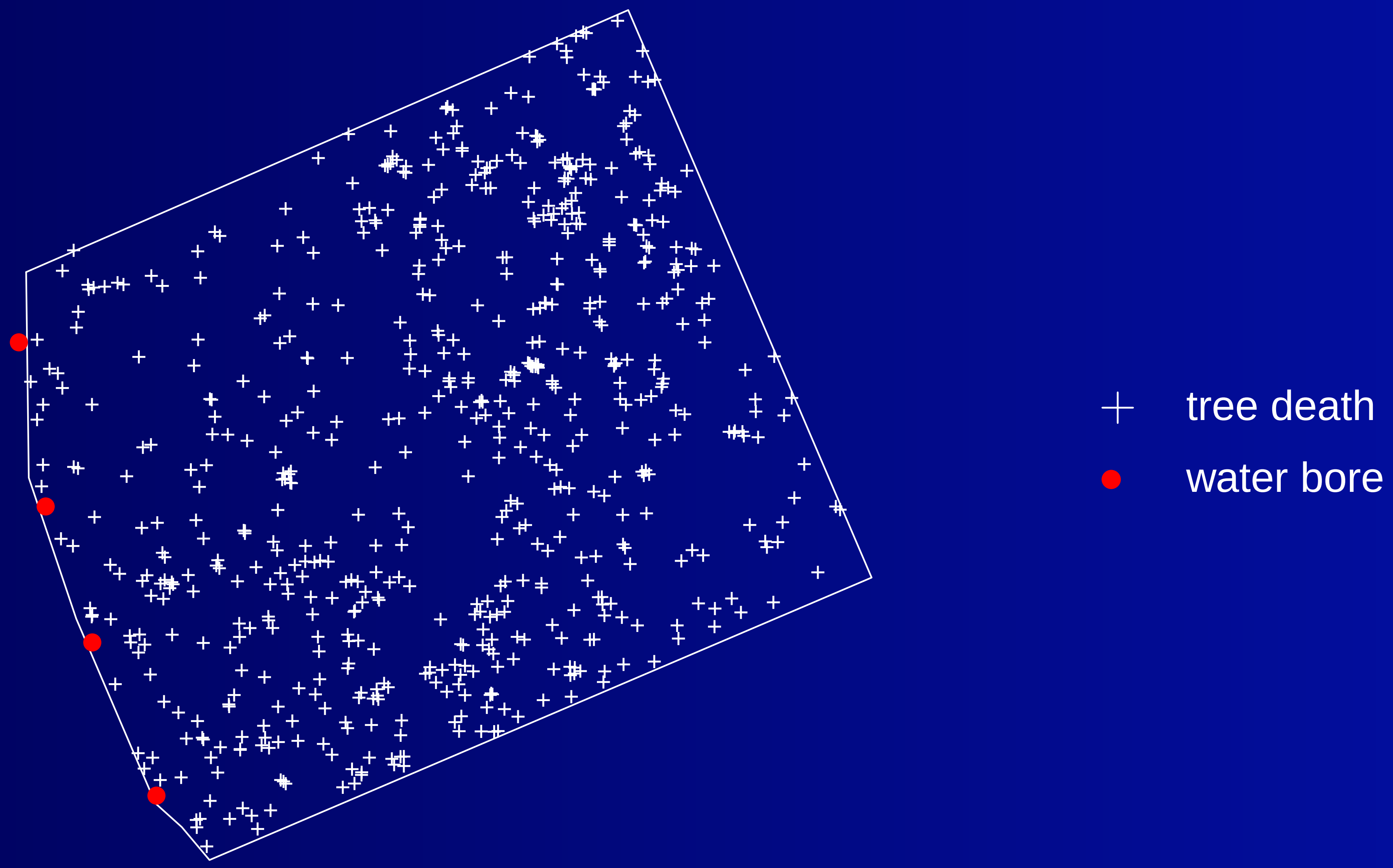
Gold in Western Australia



Galaxy survey



Tree deaths in a groundwater catchment



Common elements

- spatial locations of ‘events’/ ‘things’

  spatial point pattern

- additional data

  spatial covariates

- we want to investigate
  - dependence of points on covariate
  - dependence between points

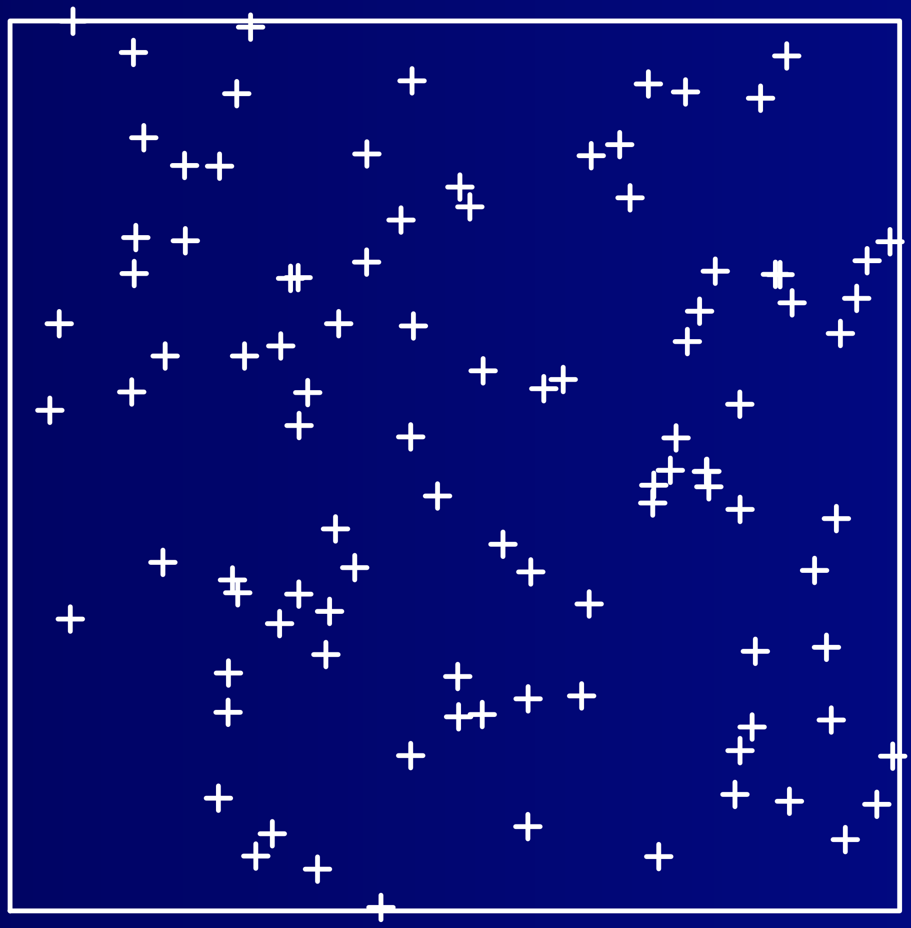
Completely random point pattern

What would

a completely random pattern

look like?

Completely random point pattern



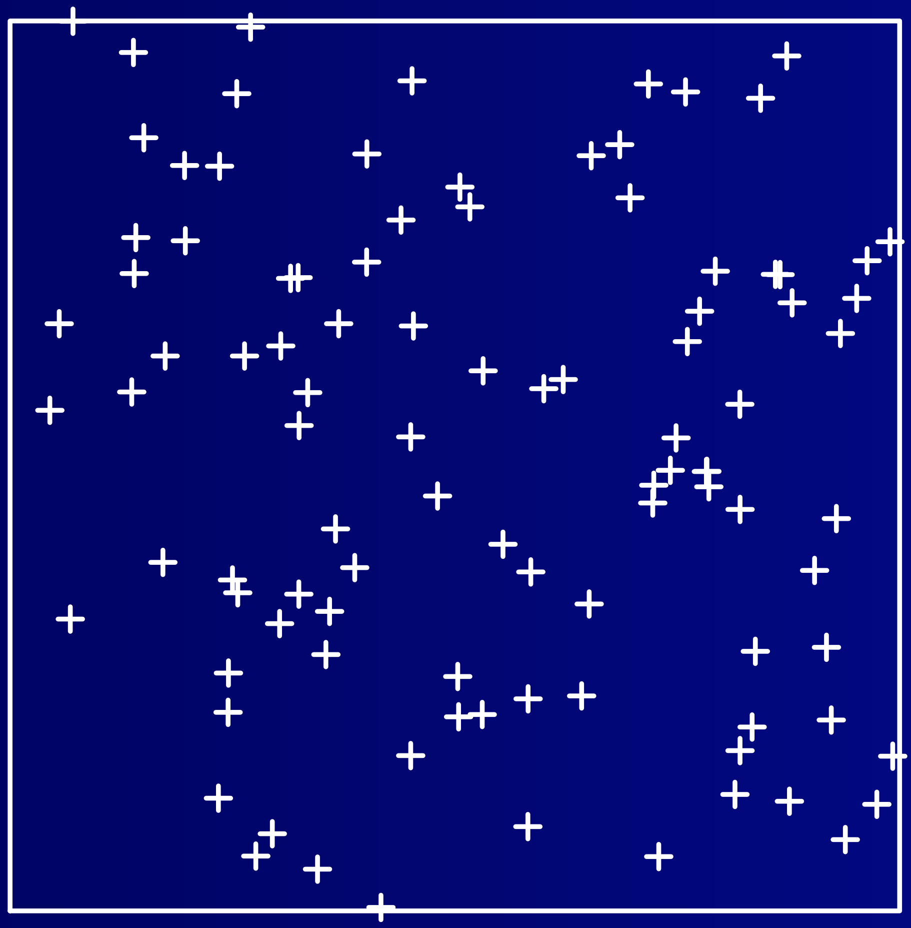
Terminology: Point process

A *point process* is a random mechanism that generates a random pattern of points.

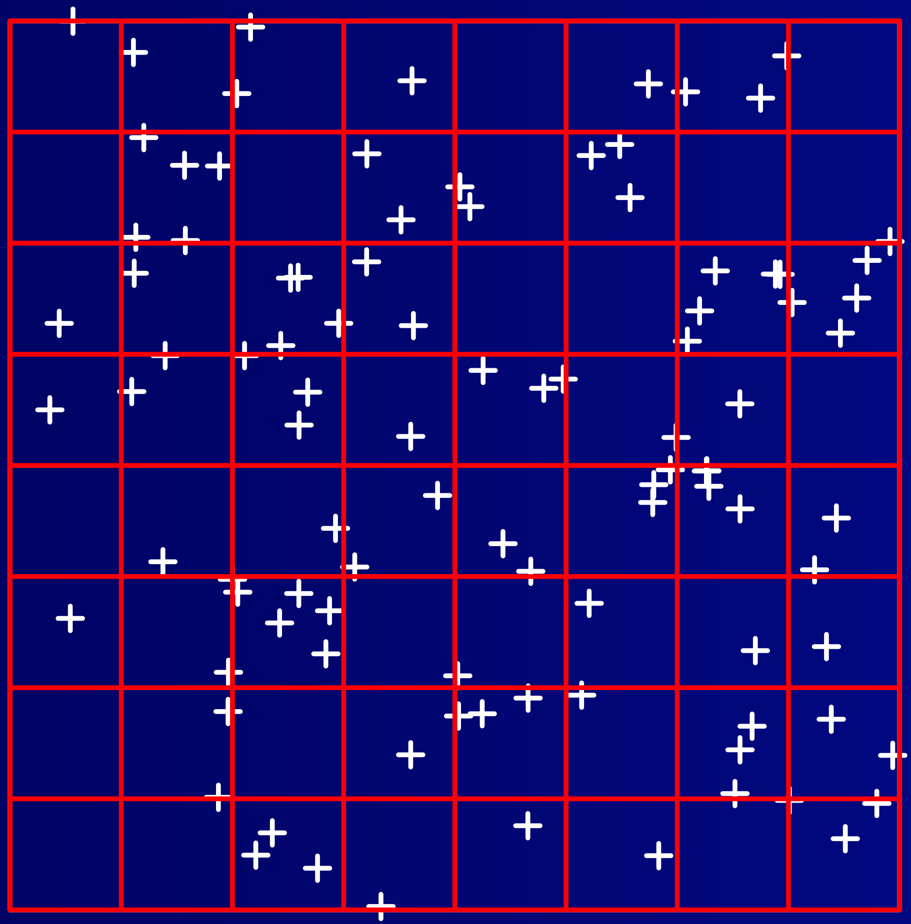
Terminology: Point process

A *point process* is any random mechanism that generates a random pattern of points.

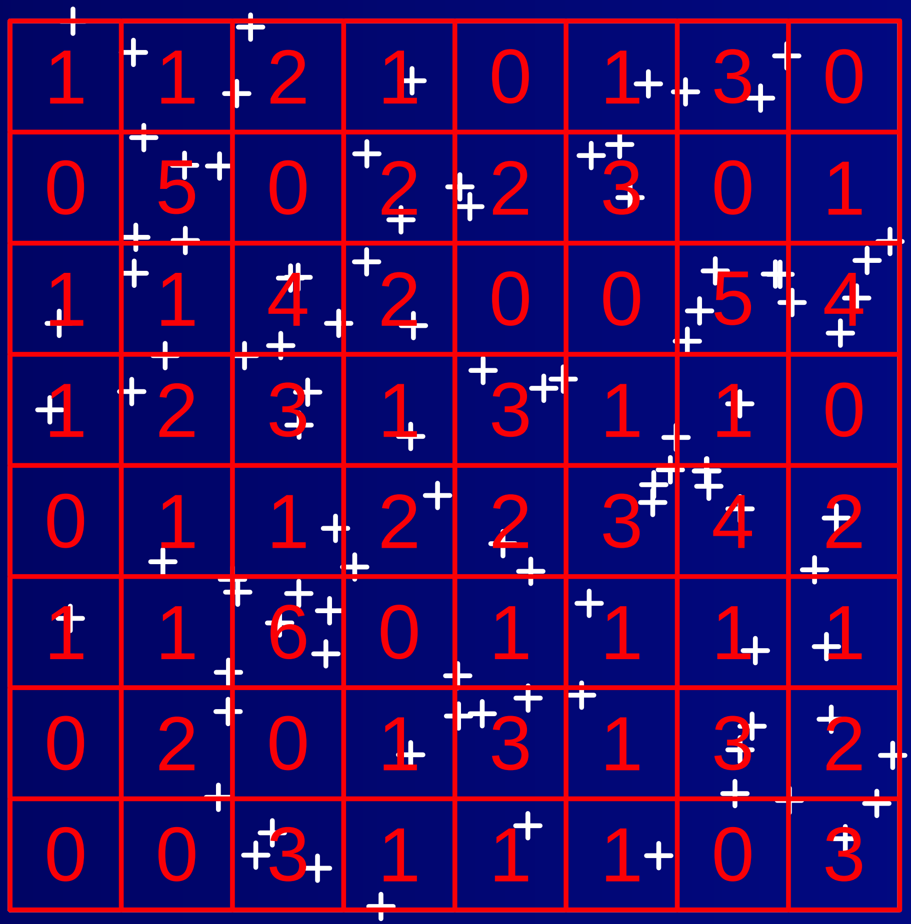
Completely random point process



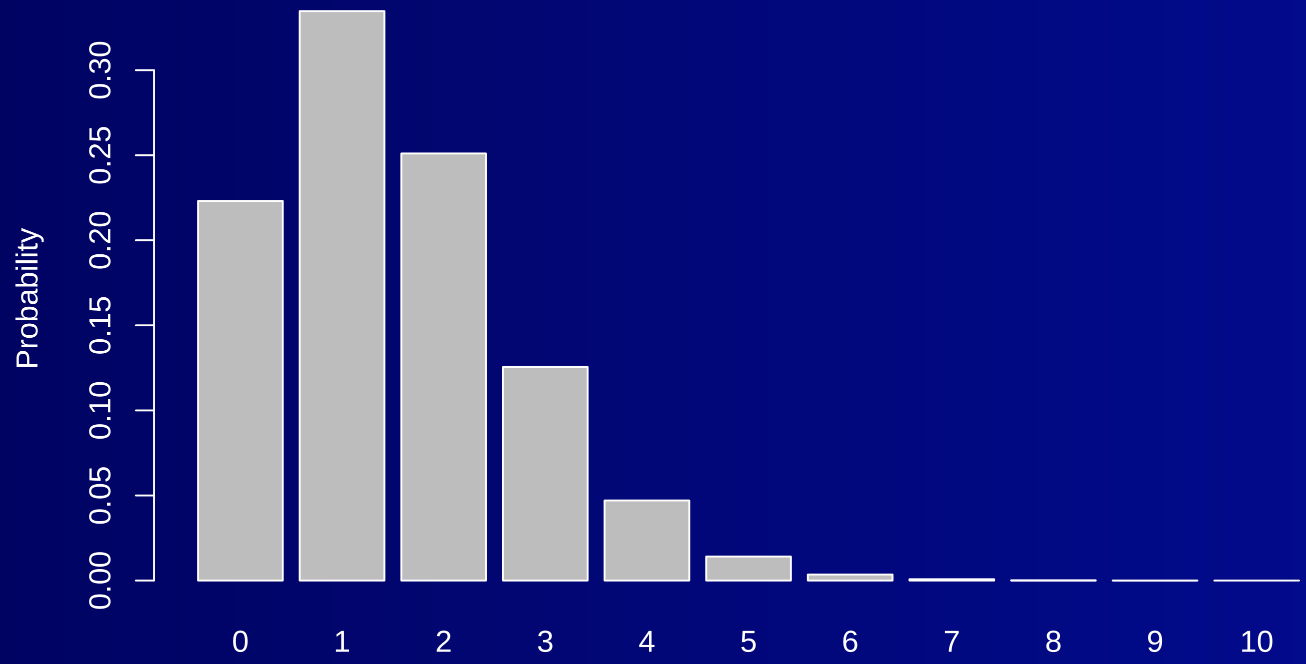
Completely random point process



Completely random point process



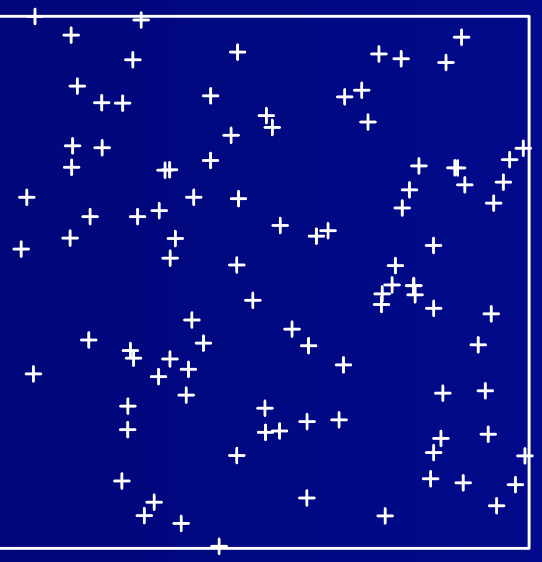
Poisson distribution (1837)



Bortkiewicz, *Das Gesetz der kleine Zahlen* 1898

Poisson point process

The canonical model for a completely random pattern of points is the *Poisson point process*

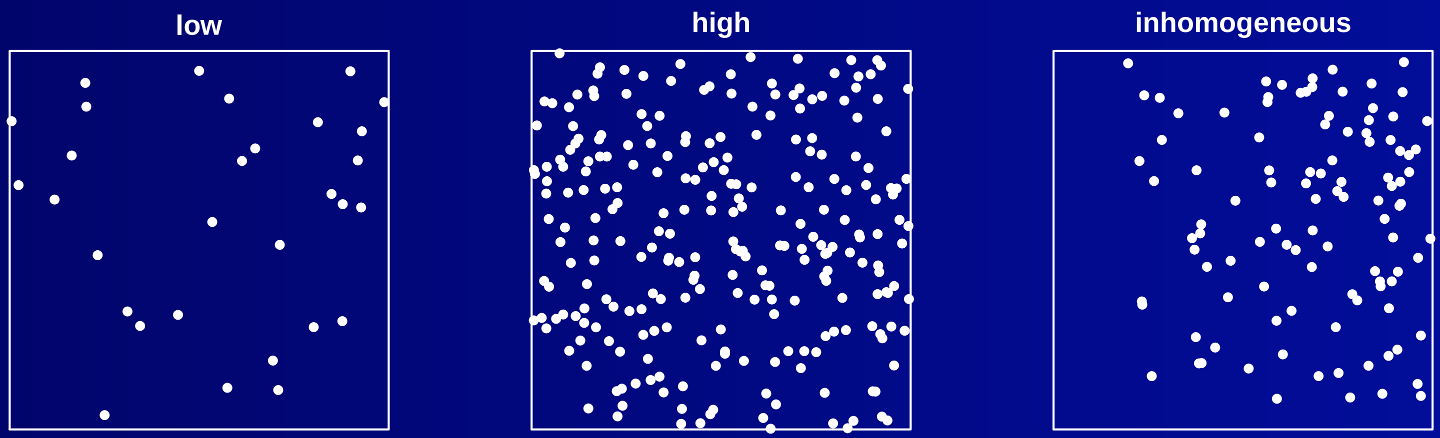


Point locations are independent of each other; different areas of the pattern are independent of each other.

Terminology: Intensity

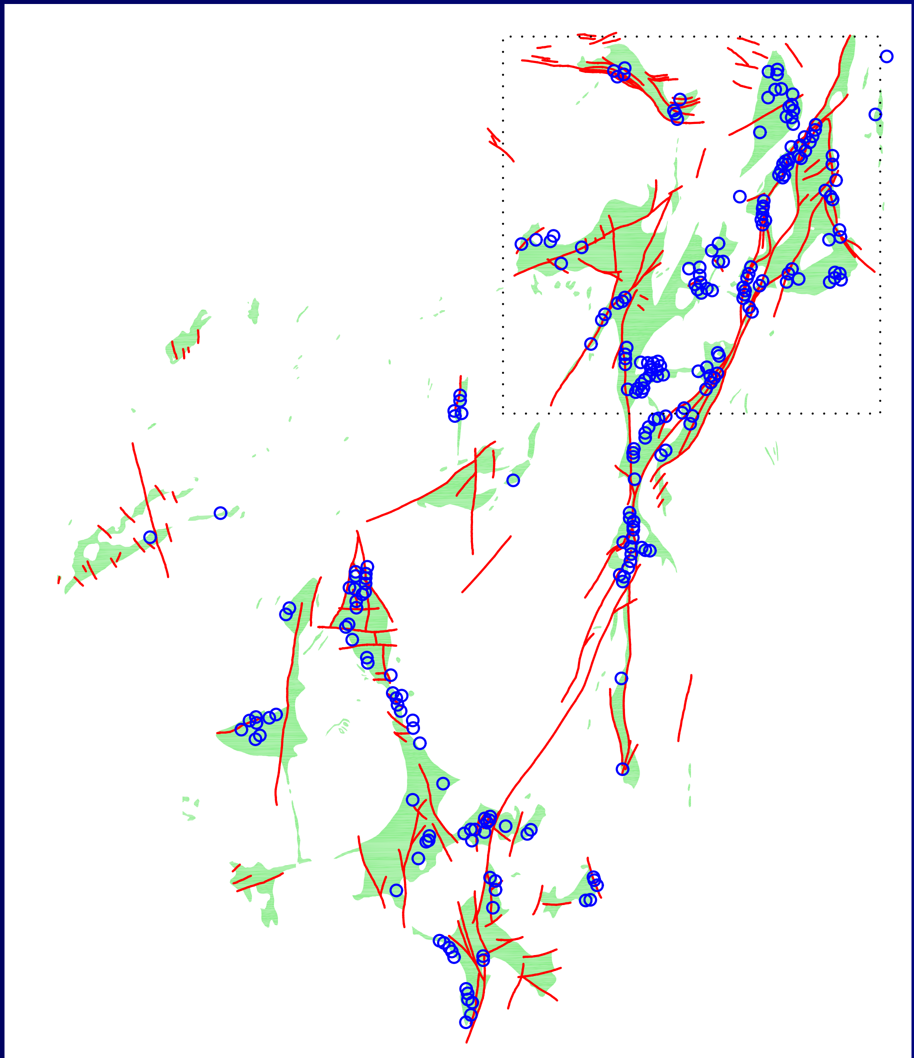
The **intensity** λ of a point process is

the expected (mean) number of points per unit area.



Intensity could be a spatially-varying function $\lambda(u)$ of location u.

Gold in Western Australia



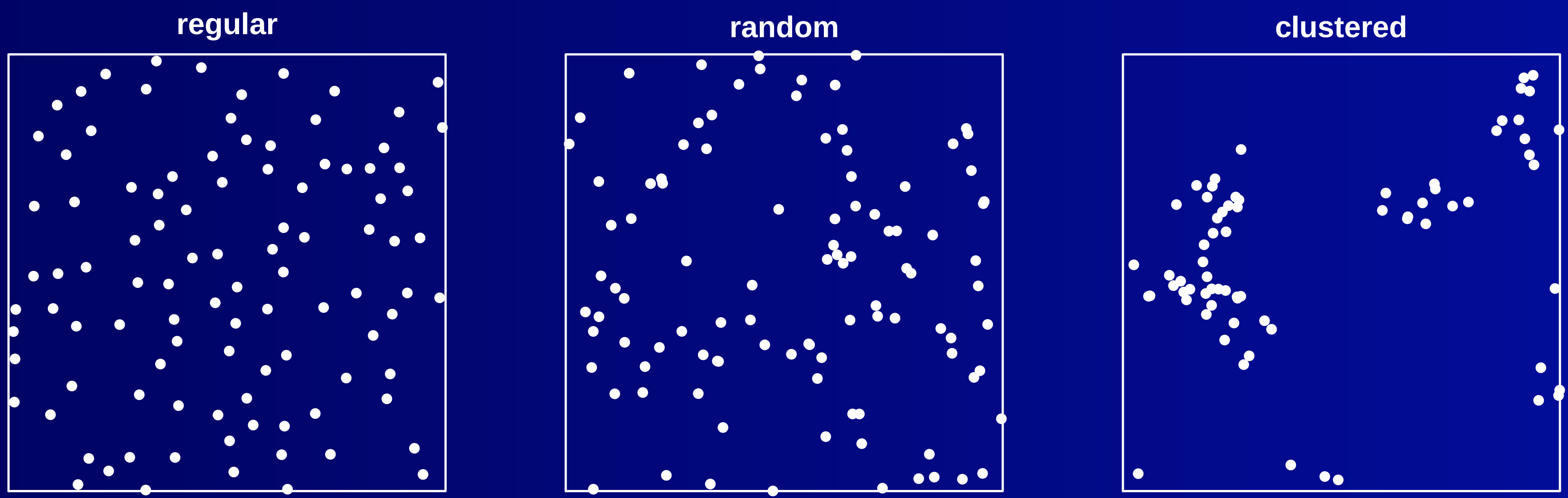
Problem 1: Intensity

Investigate whether the intensity λ depends on the spatial covariate Z:

$$\lambda(u) = f(Z(u))$$

Terminology: Interaction

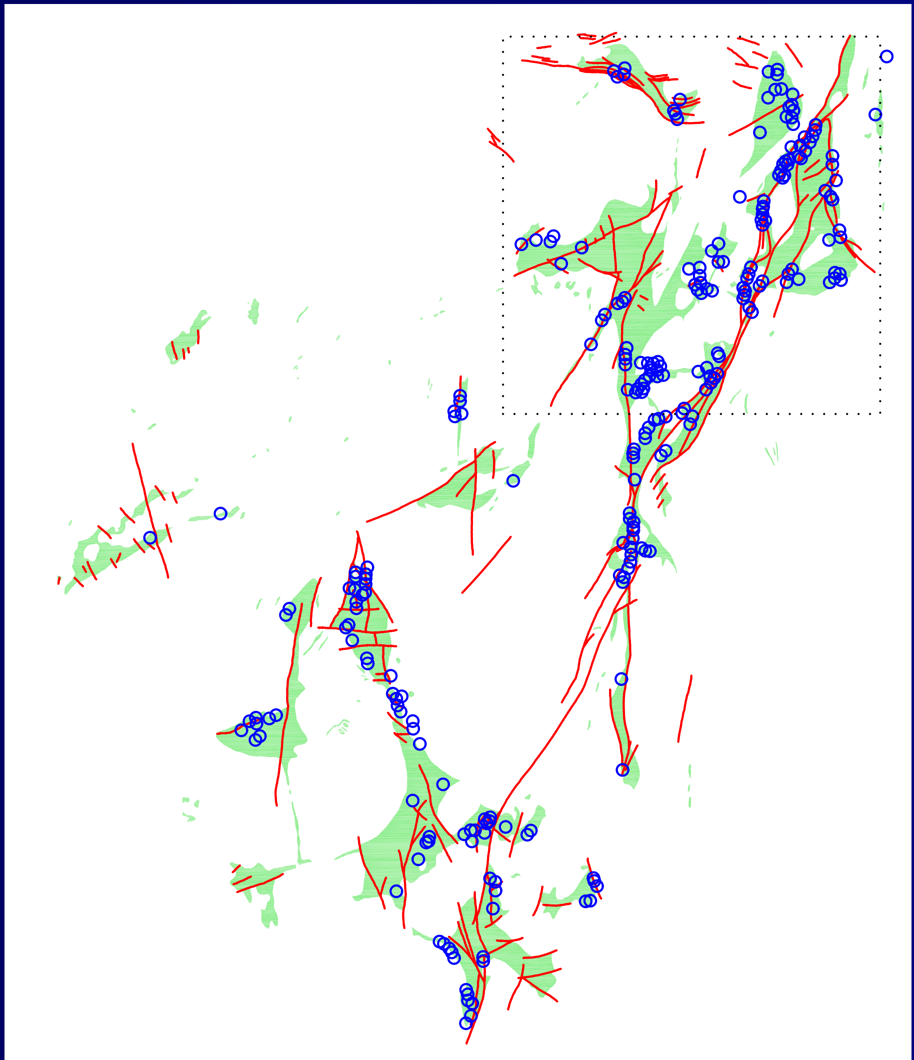
Interpoint interaction in a point process is stochastic dependence between the locations of the points.



Problem 2: Interaction

Detect and describe interaction between the points of a point process, after allowing for variation in intensity.

Gold in Western Australia

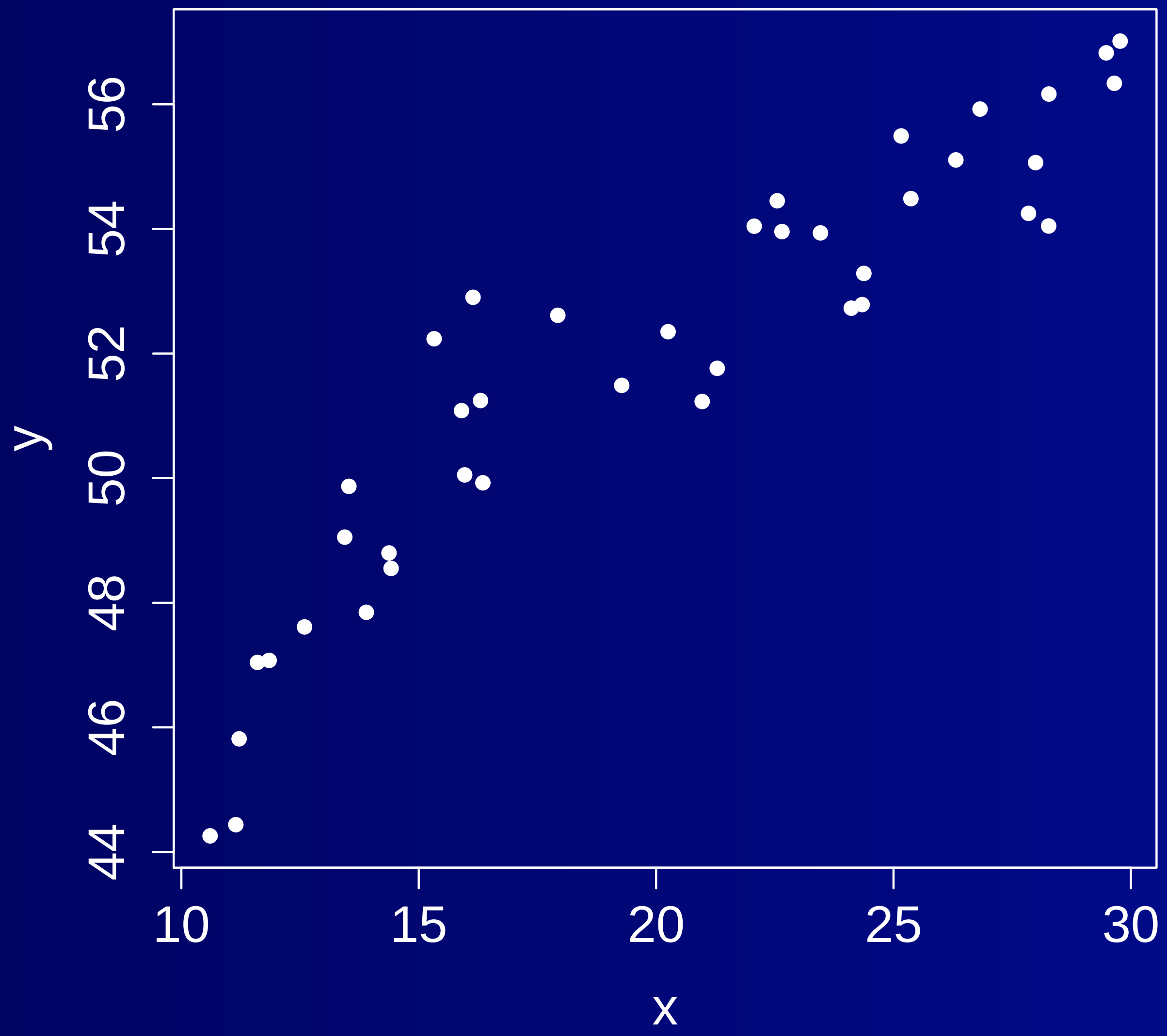


Problem 1: Intensity

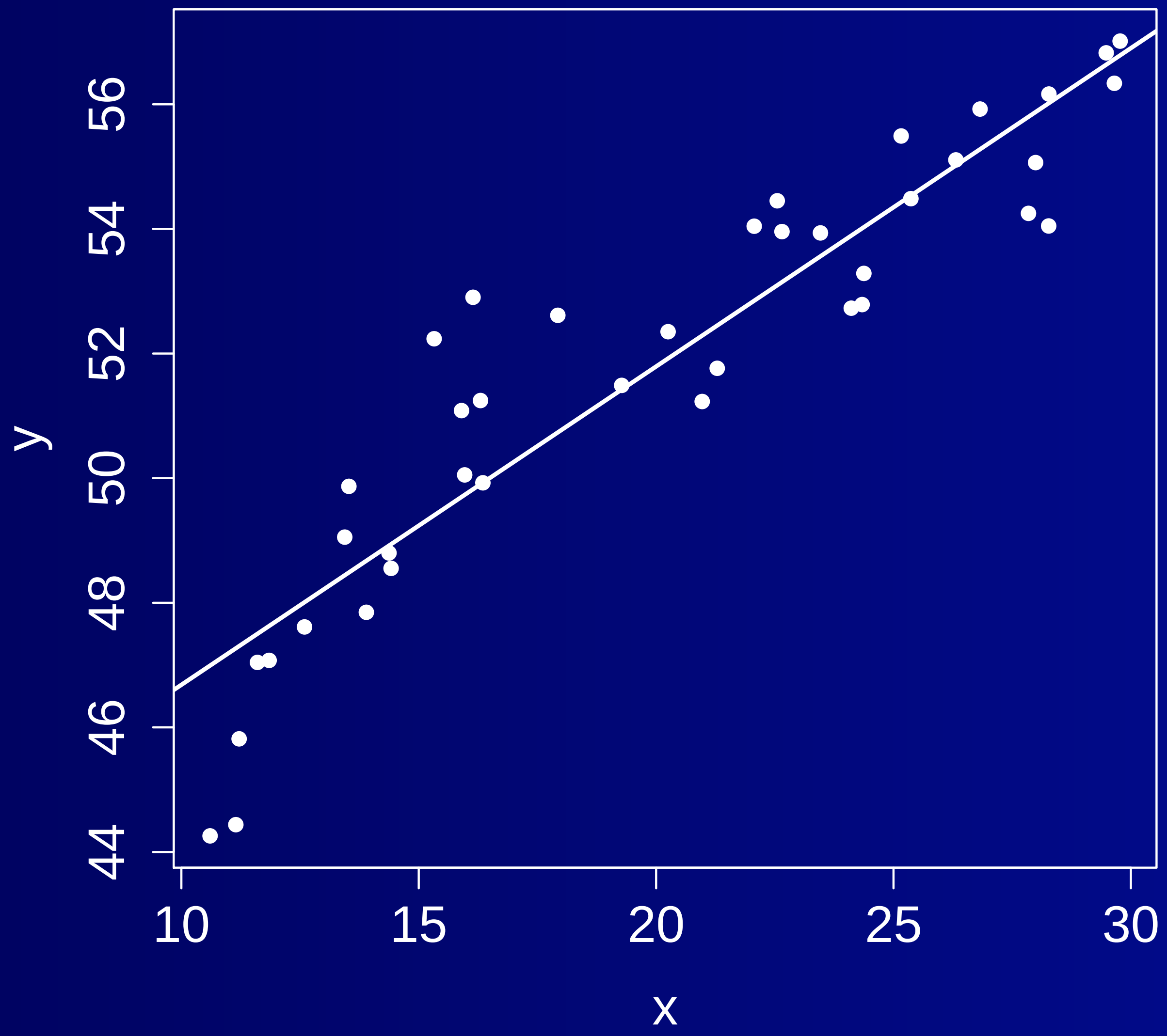
Investigate whether the intensity λ depends on the spatial covariate Z:

$$\lambda(u) = f(Z(u))$$

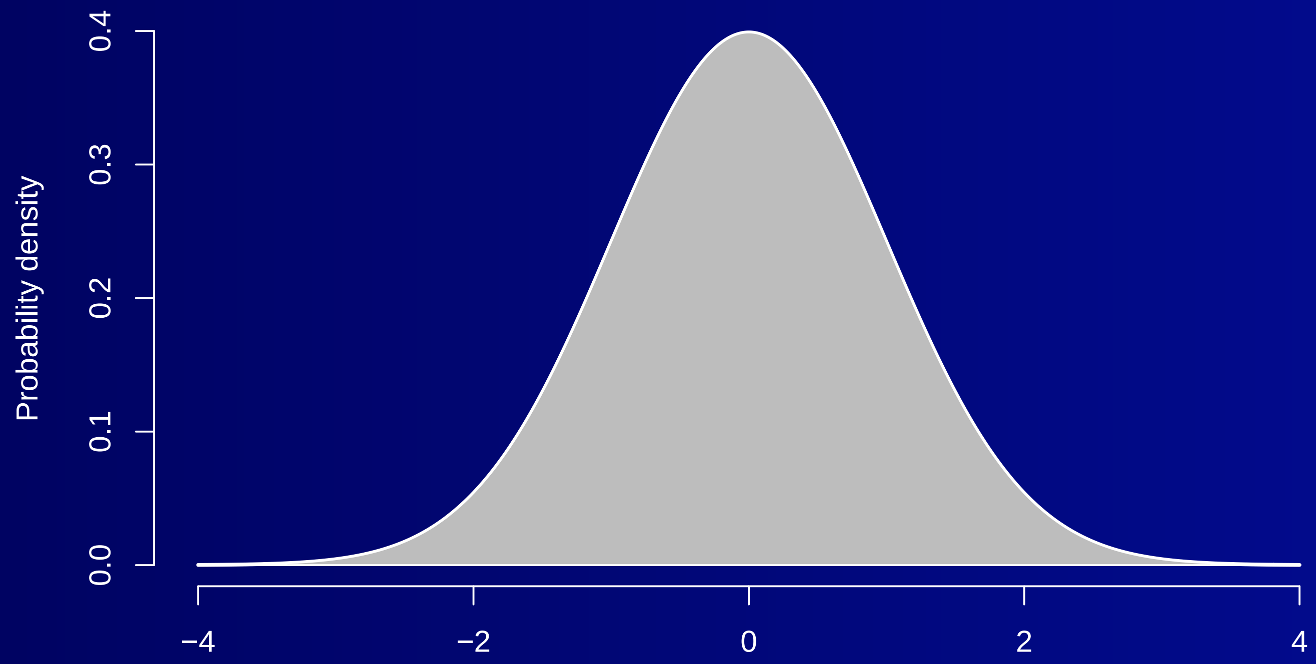
Regression



Linear regression

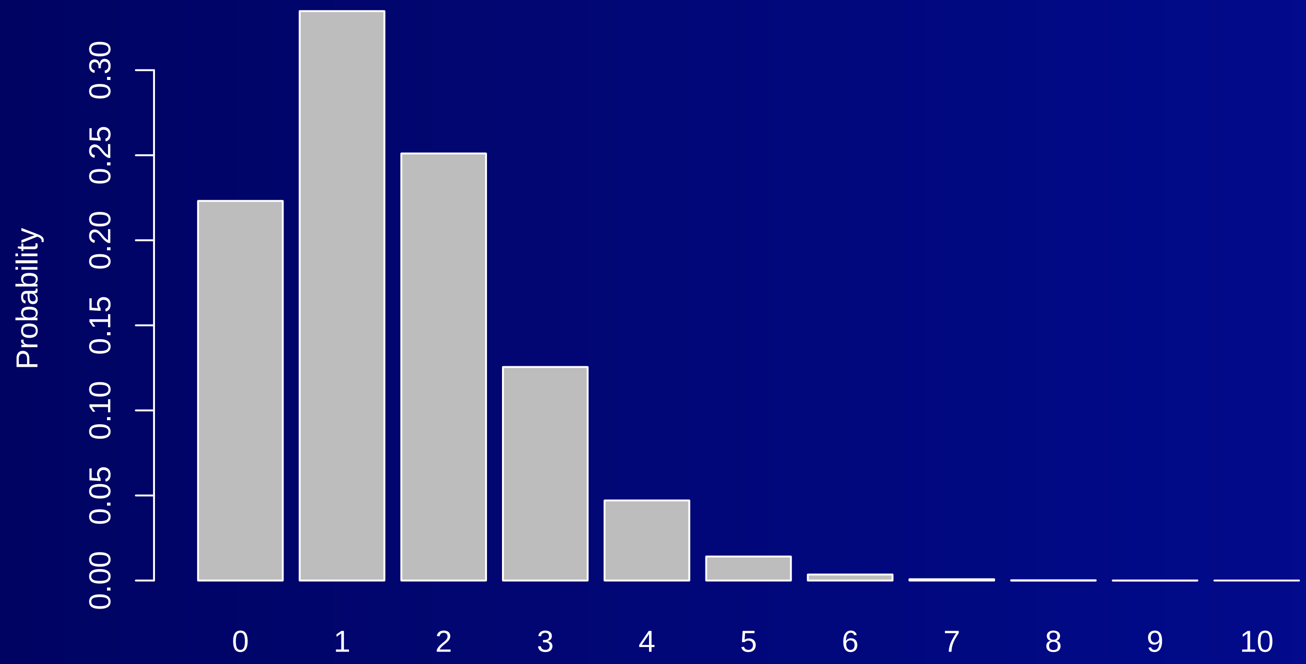


Normal distribution (1809)

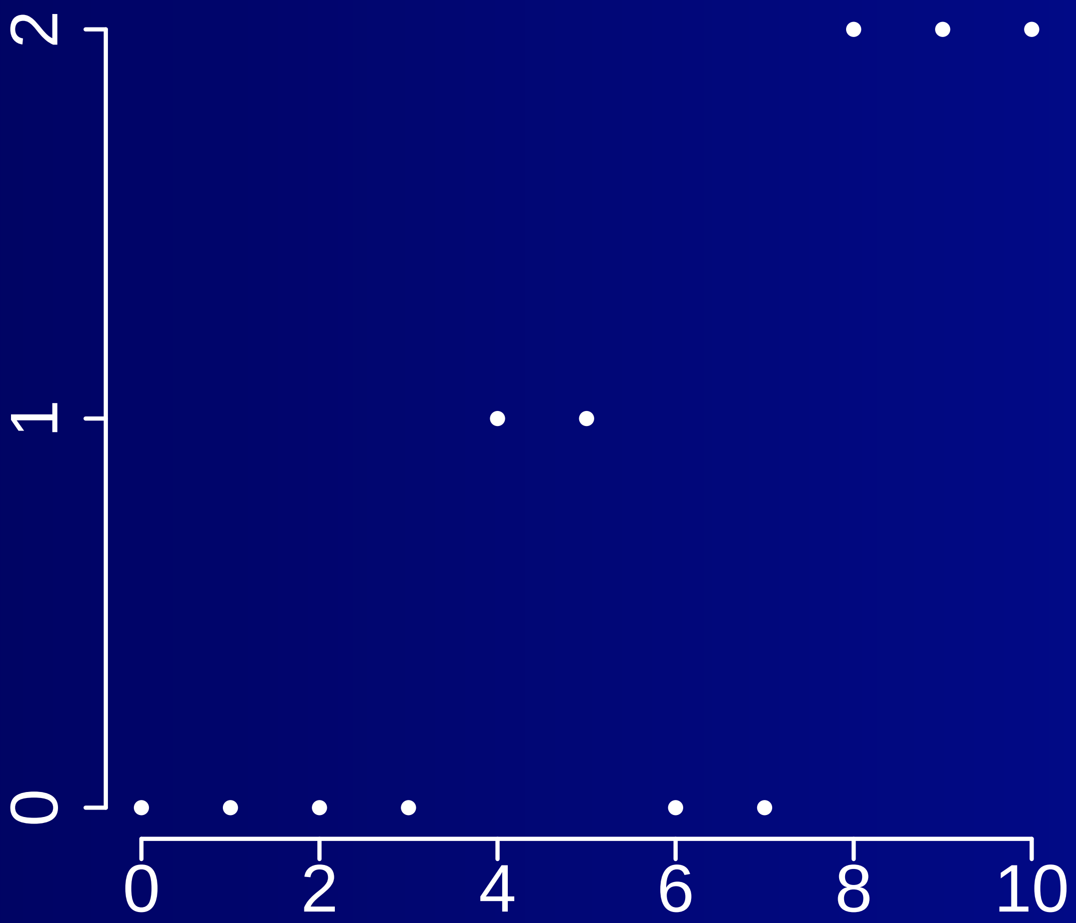


The standard method for linear regression ('least squares') is appropriate when the errors around the line are Normally distributed.

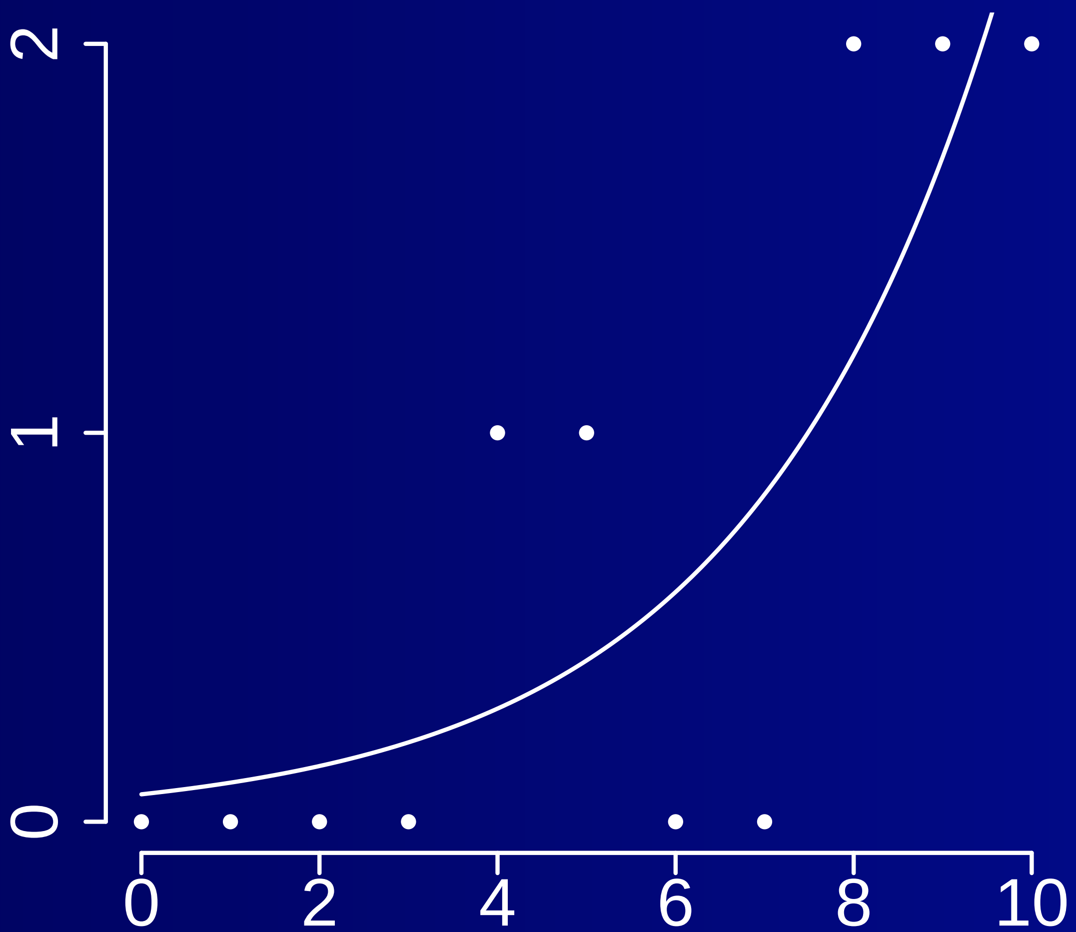
Poisson distribution



Poisson regression



Poisson regression



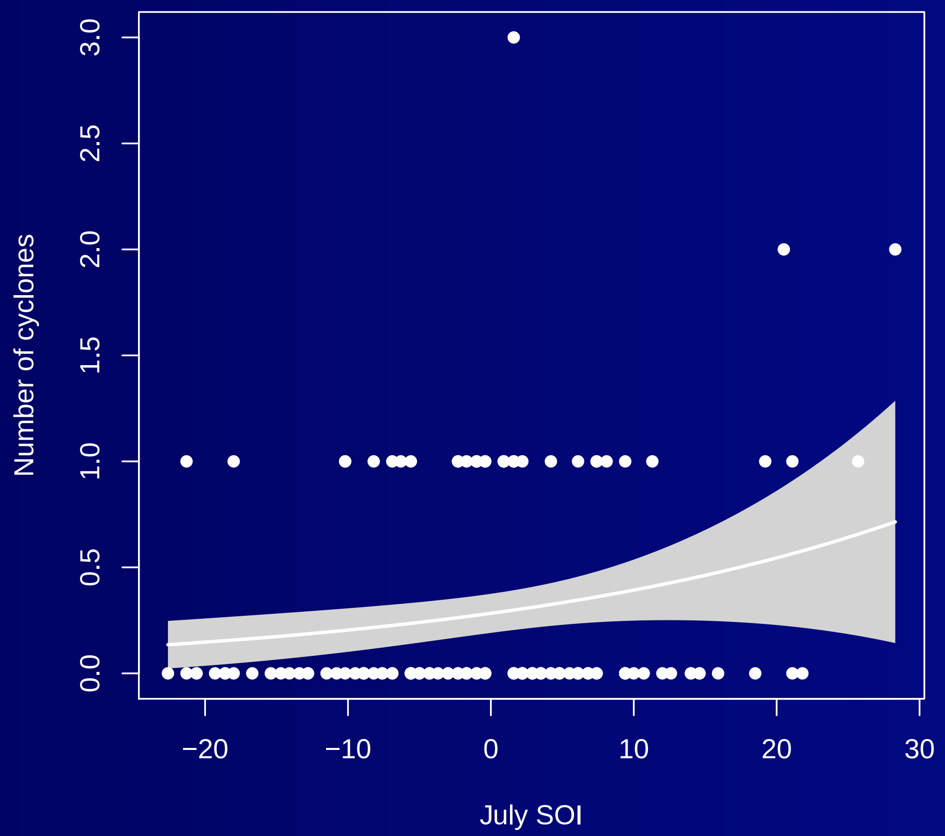
Generalized Linear Models

The theory of “generalized linear models” embraces

- linear regression
- Poisson regression
- ...

$\boxed{\text{This unification was achieved in 1980}}$

Poisson regression



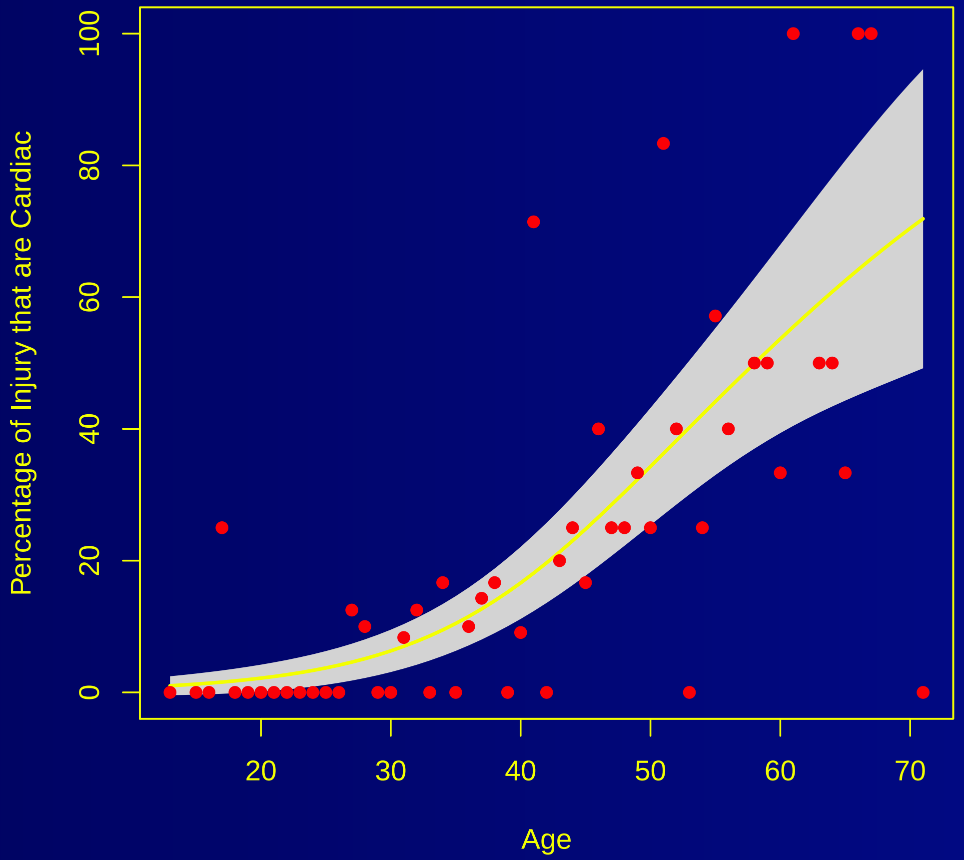
Severe cyclones

in Australia

regression on

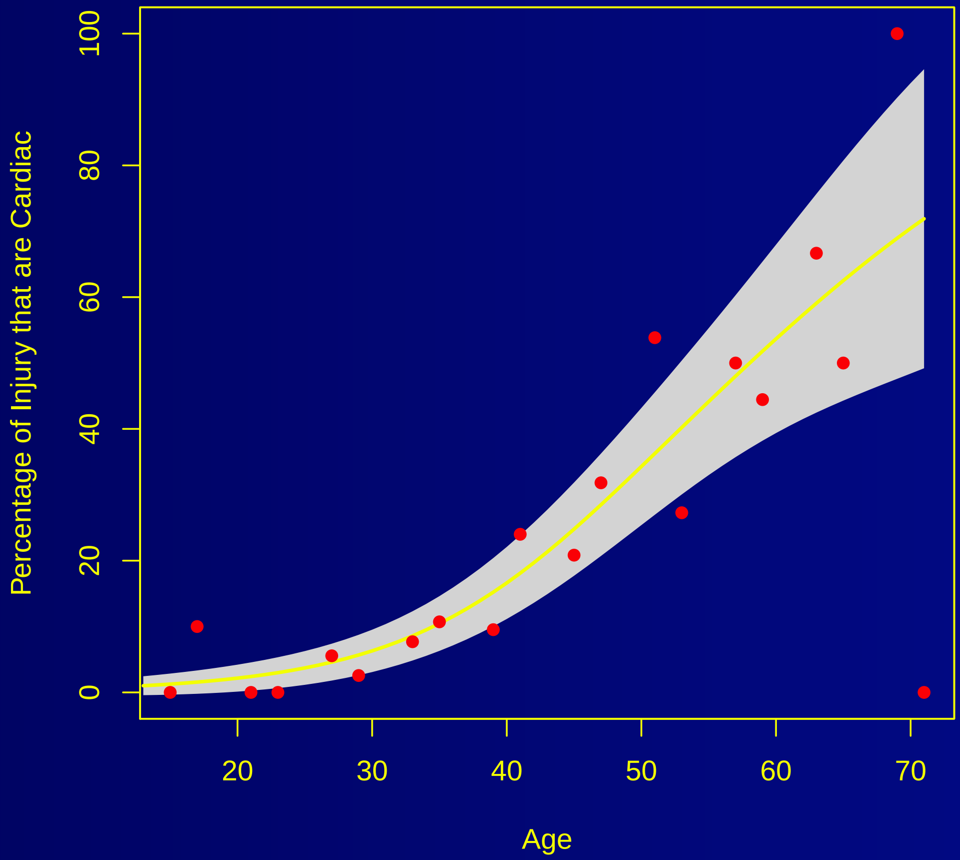
El Niño

Poisson regression



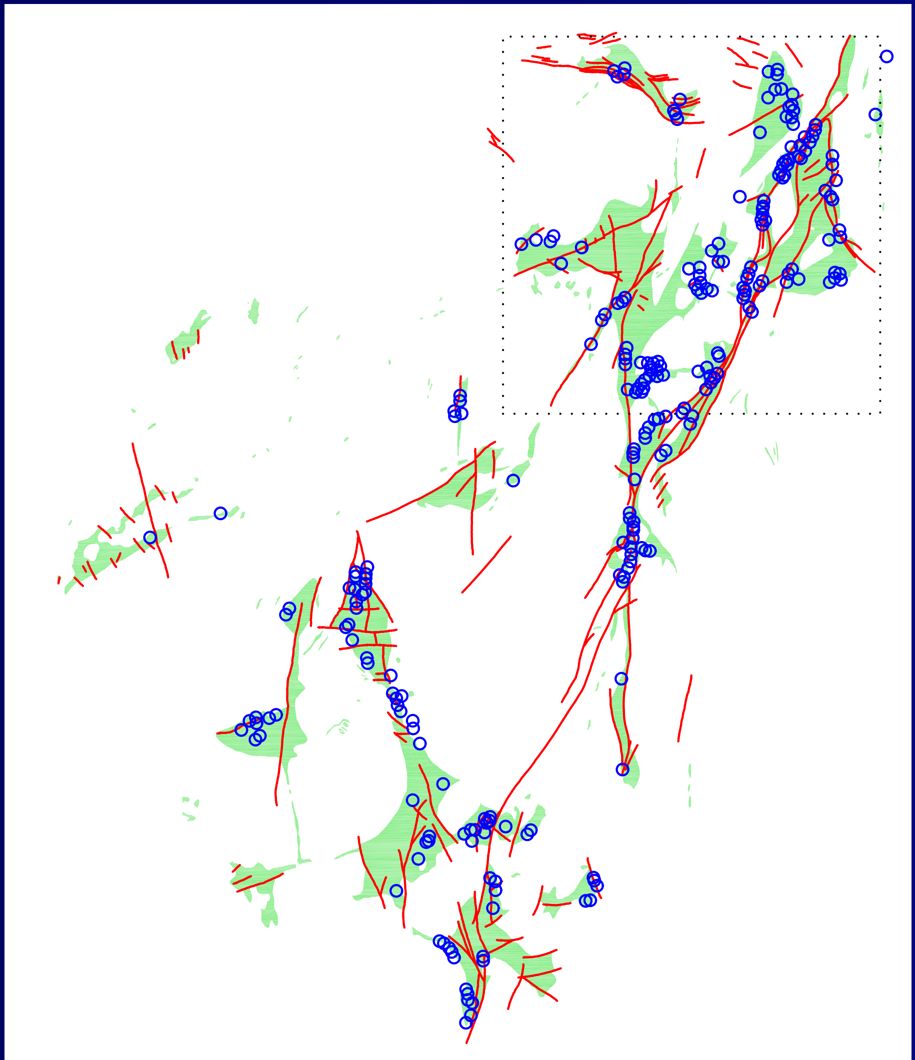
Scuba diving deaths in Australia

Poisson regression



3-year age groups

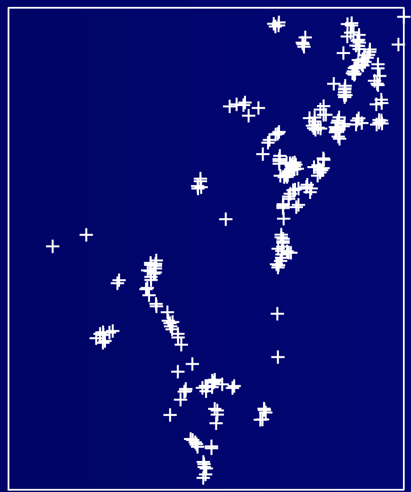
Gold deposits



Pixel Logistic Regression

Proposed by statistician John Tukey 1972, developed by geologist Frits Agterberg

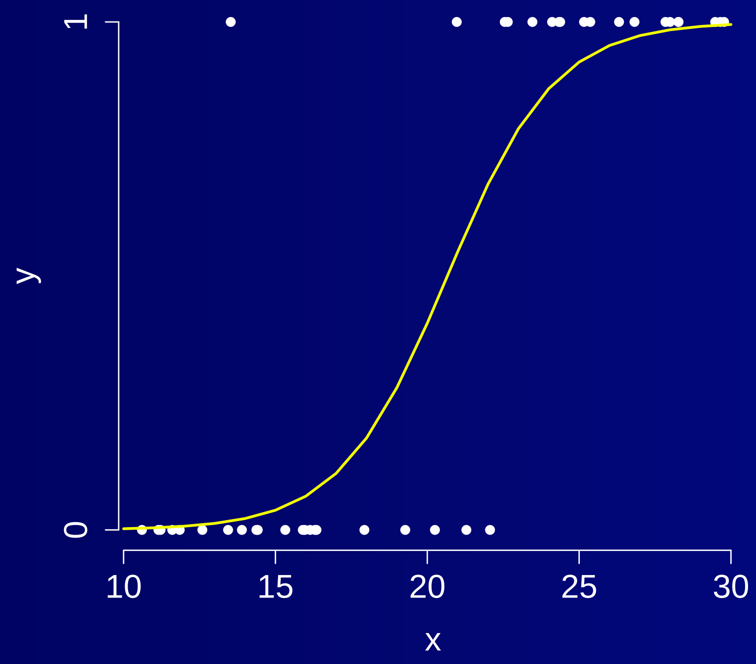
- Divide survey region into pixels
- Set pixel value to 1 if it contains data points, otherwise 0



0	0	0	0	0	0	0	0	0	0	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1
0	0	0	0	0	0	0	0	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	1	1	1	1	0	0
0	0	0	0	0	0	0	1	0	0	1	1	0	0	0
0	0	0	0	0	0	0	0	1	0	1	1	0	0	0
0	1	1	0	0	1	0	0	0	0	1	0	0	0	0
0	0	0	0	1	1	0	0	0	0	1	0	0	0	0
0	0	0	0	0	1	1	0	0	0	1	0	0	0	0
0	0	0	1	0	0	1	0	0	0	1	0	0	0	0
0	0	0	0	0	0	1	1	1	0	0	0	0	0	0
0	0	0	0	0	0	1	1	0	1	0	0	0	0	0
0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0

- Analyse 0/1 pixel values using logistic regression

Logistic regression

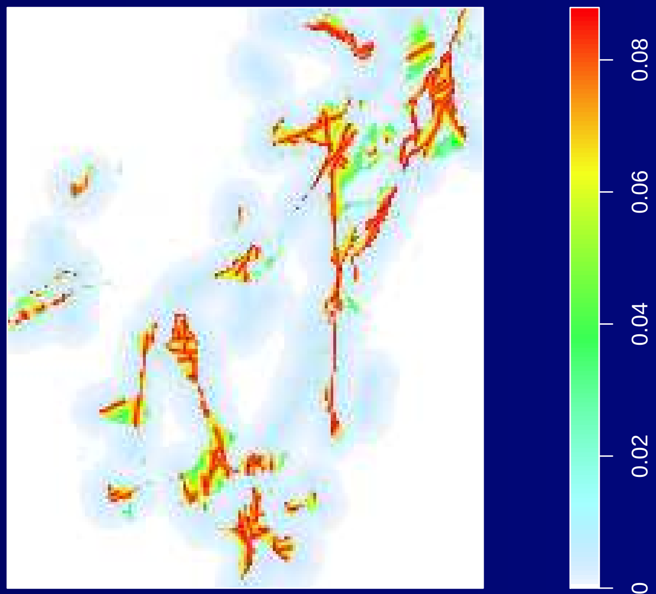


Logistic regression

$$\log \frac{p}{1-p} = \beta_0 + \beta_1 x$$

Gold deposits

Logistic regression analysis



Predicted probability of a gold deposit

in each 2×2 km pixel

Logistic regression of

$y =$ presence/absence of gold

on

$x =$ distance to nearest fault

Claims about pixel logistic regression

- "logistic regression is a nonparametric technique"

  (i.e. does not make assumptions about the relation between y and x)

- results depend on choice of pixel size

- "difficult to interpret the fitted parameters"

- small pixels $\Rightarrow$ numerical problems

Those who ignore statistics

are doomed to reinvent it.

— B. Efron

Effect of pixel size

- results using different pixel sizes are **incompatible**
- there is no point process in continuous space that is consistent with logistic regression on every pixel grid

Baddeley et al, Spatial logistic regression and change-of-support for Poisson point processes. *Electronic Journal of Statistics* **4** (2010)

Very small pixels

For very small pixel size, pixel logistic regression is equivalent to assuming a **Poisson point process** with loglinear intensity

$$\lambda(u) = \exp(\beta_0 + \beta_1 X(u))$$

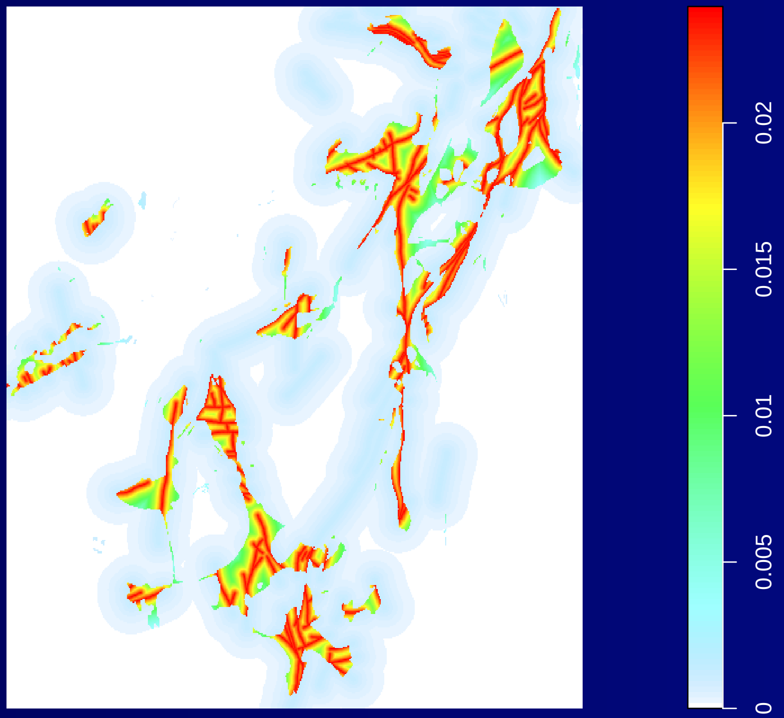
where $X(u)$ is the covariate value at spatial location u.

Warton & Shepherd, Poisson point process models solve the “pseudo-absence problem” for presence-only data in ecology. *Annals of Applied Statistics* **4** (2010)

Baddeley et al, Spatial logistic regression and change-of-support for Poisson point processes. *Electronic Journal of Statistics* **4** (2010)

Gold deposits

Loglinear Poisson point process model



Predicted intensity of gold deposits

(number of deposits per km^2)

Predictions about a Poisson point process

For a Poisson point process with given parameters, it is straightforward to predict

- expected number of deposits in a given area
- probability of exactly n deposits in a given area
- probability of finding at least one deposit within x km of a given place
- etc

20th Century statistical methodology for point patterns

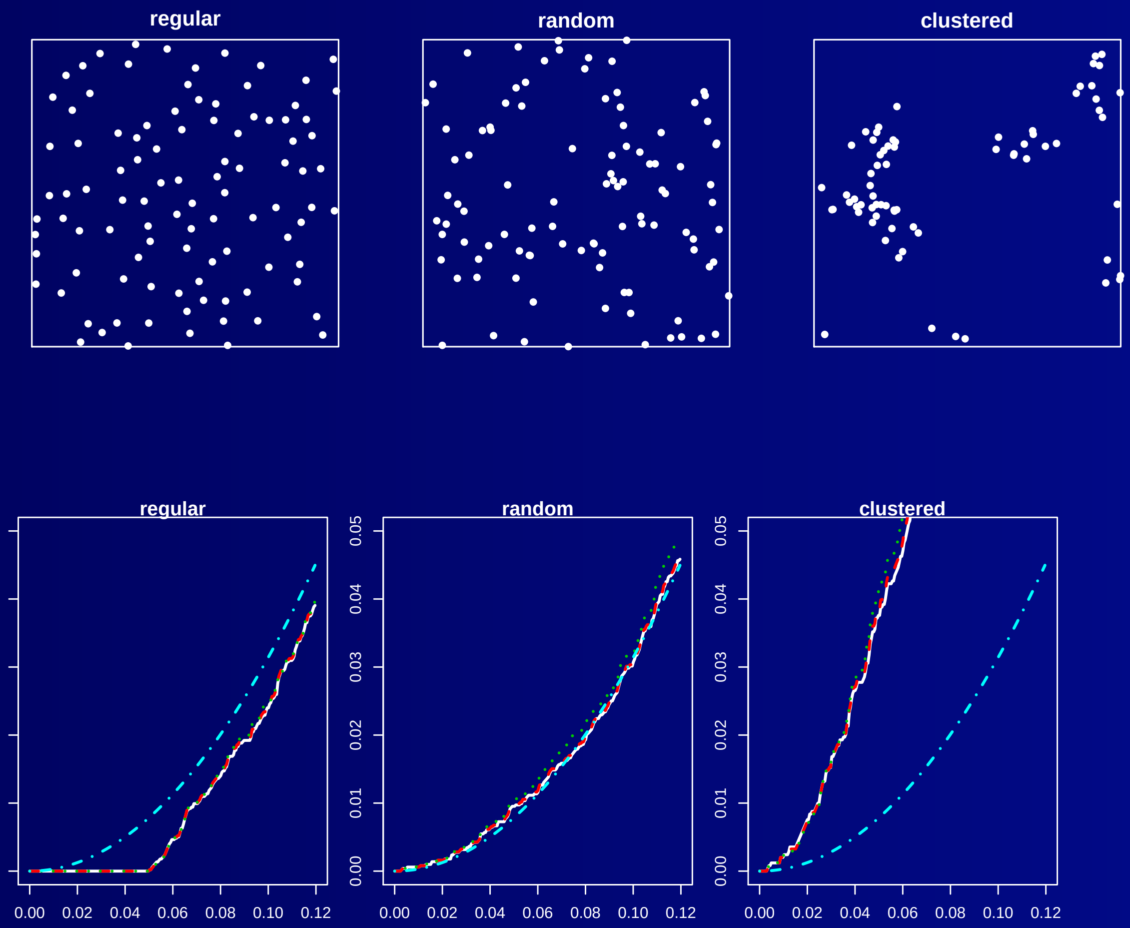
Until 1990 it was widely believed that mainstream statistical methods (‘maximum likelihood’) were “infeasible” for spatial point patterns, if interaction is present.

Instead, new methods were developed

- Markov chain Monte Carlo
- composite likelihood
- moment methods

20th Century statistical methodology for point patterns

Moment methods: Ripley's K function



20th Century statistical methodology for point patterns

Critique

- Clunky

  Inflexible, slow, temperamental

- Doesn't answer real world questions

  e.g. "How confident are you that there is gold in this region?"

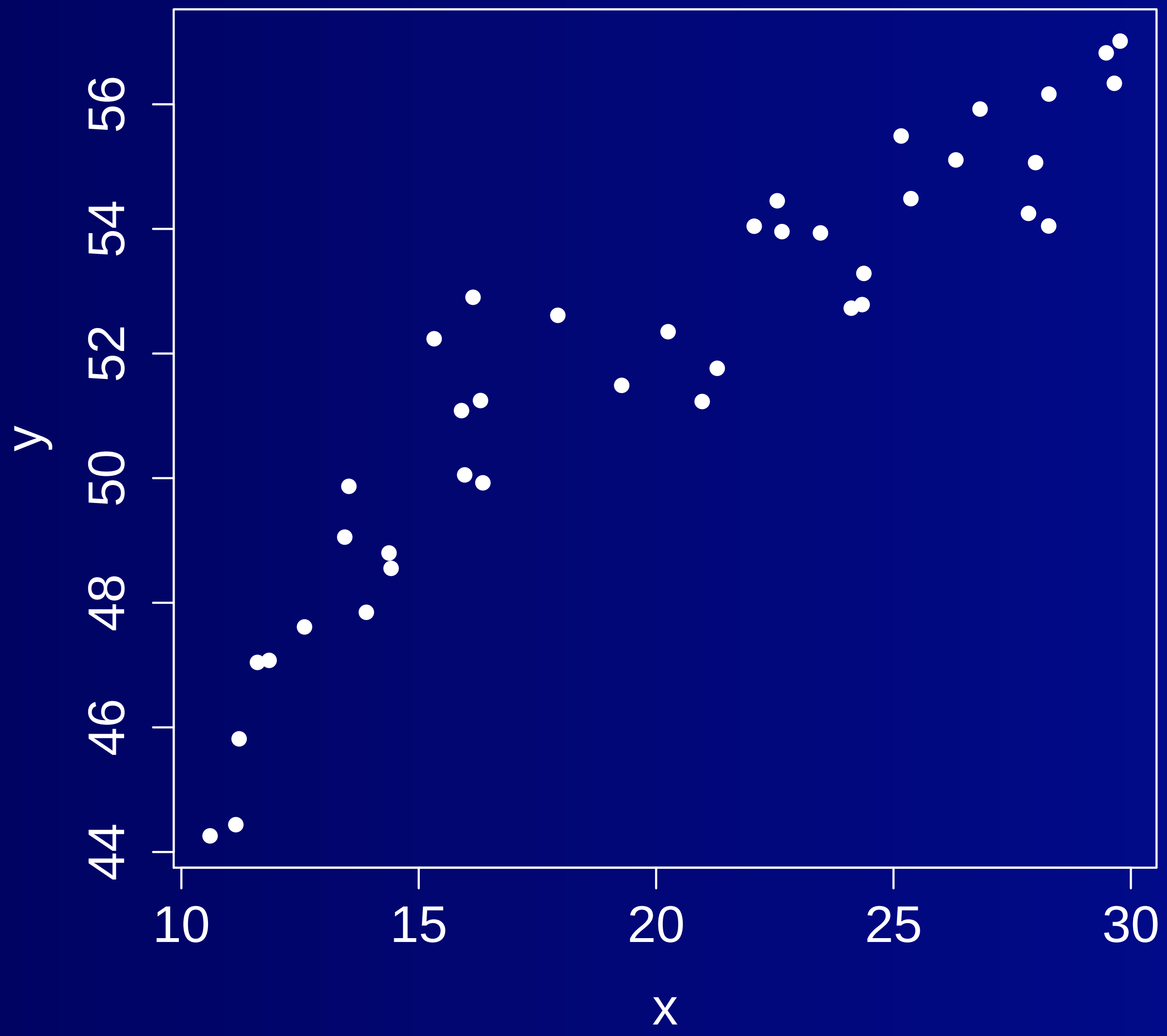
- Immature

  Doesn't provide standard statistical tools e.g. confidence interval, goodness-of-fit, leverage, influence, residuals, partial residuals

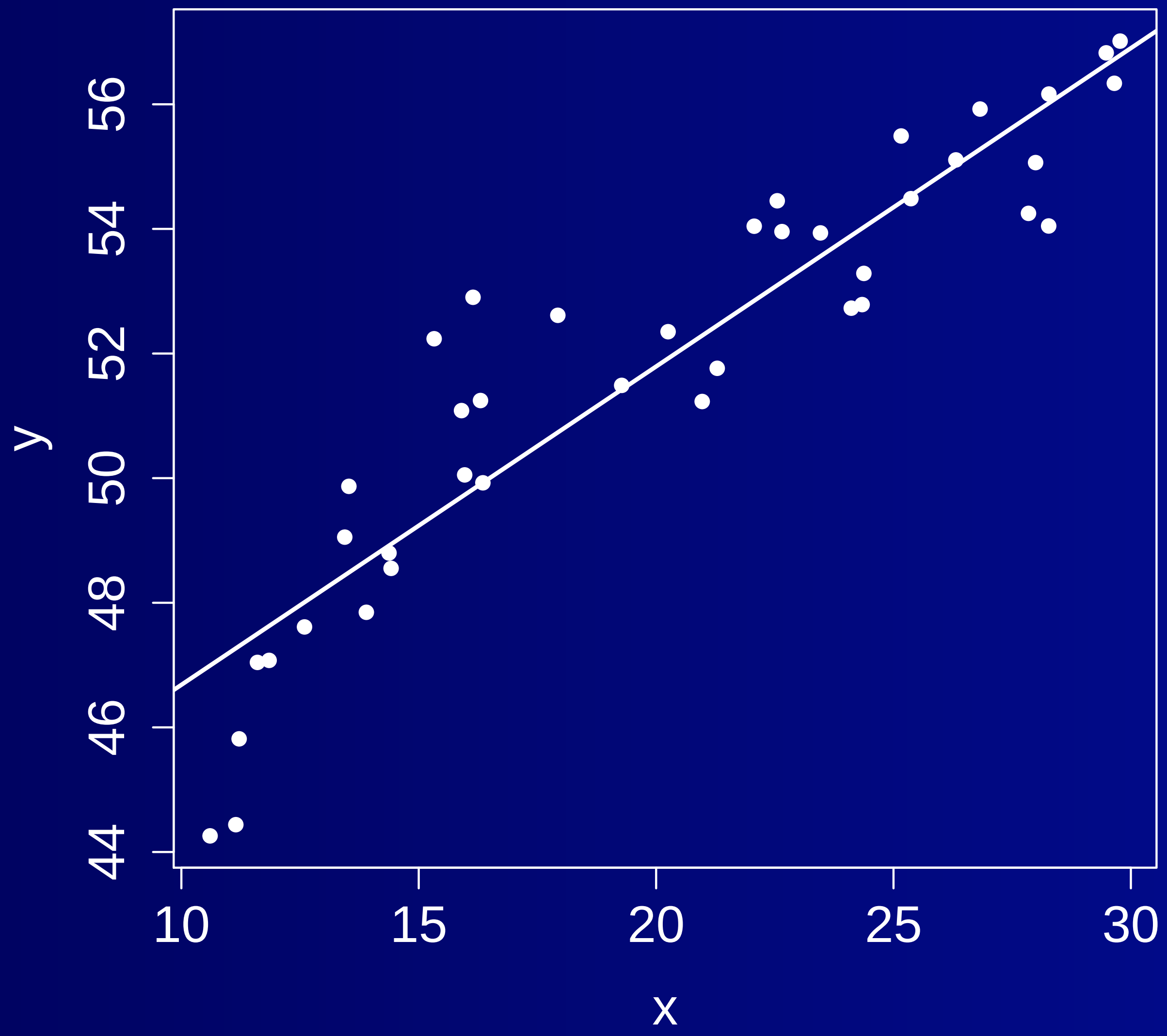
Statistical tools

Statistical methodology

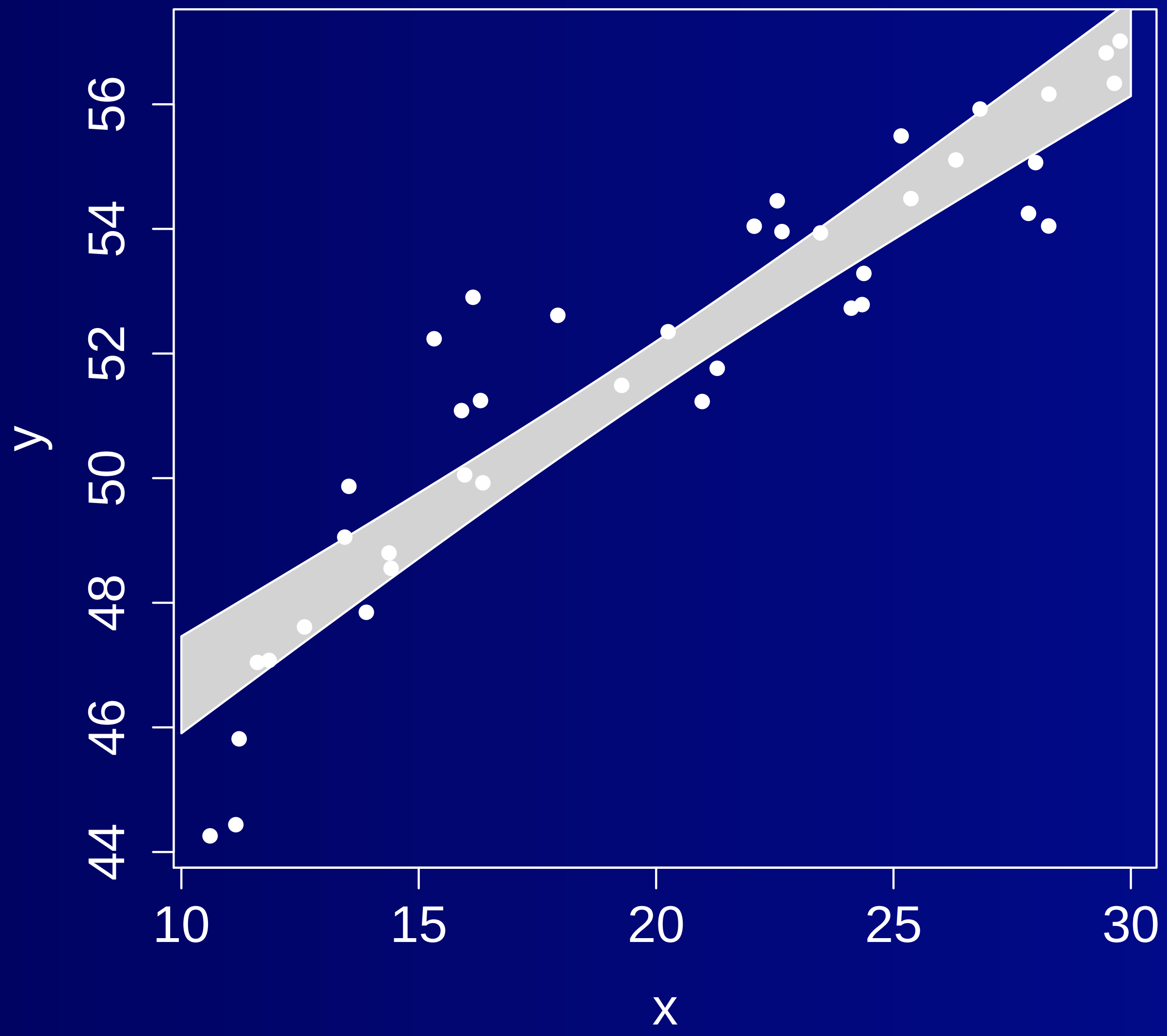
Linear regression



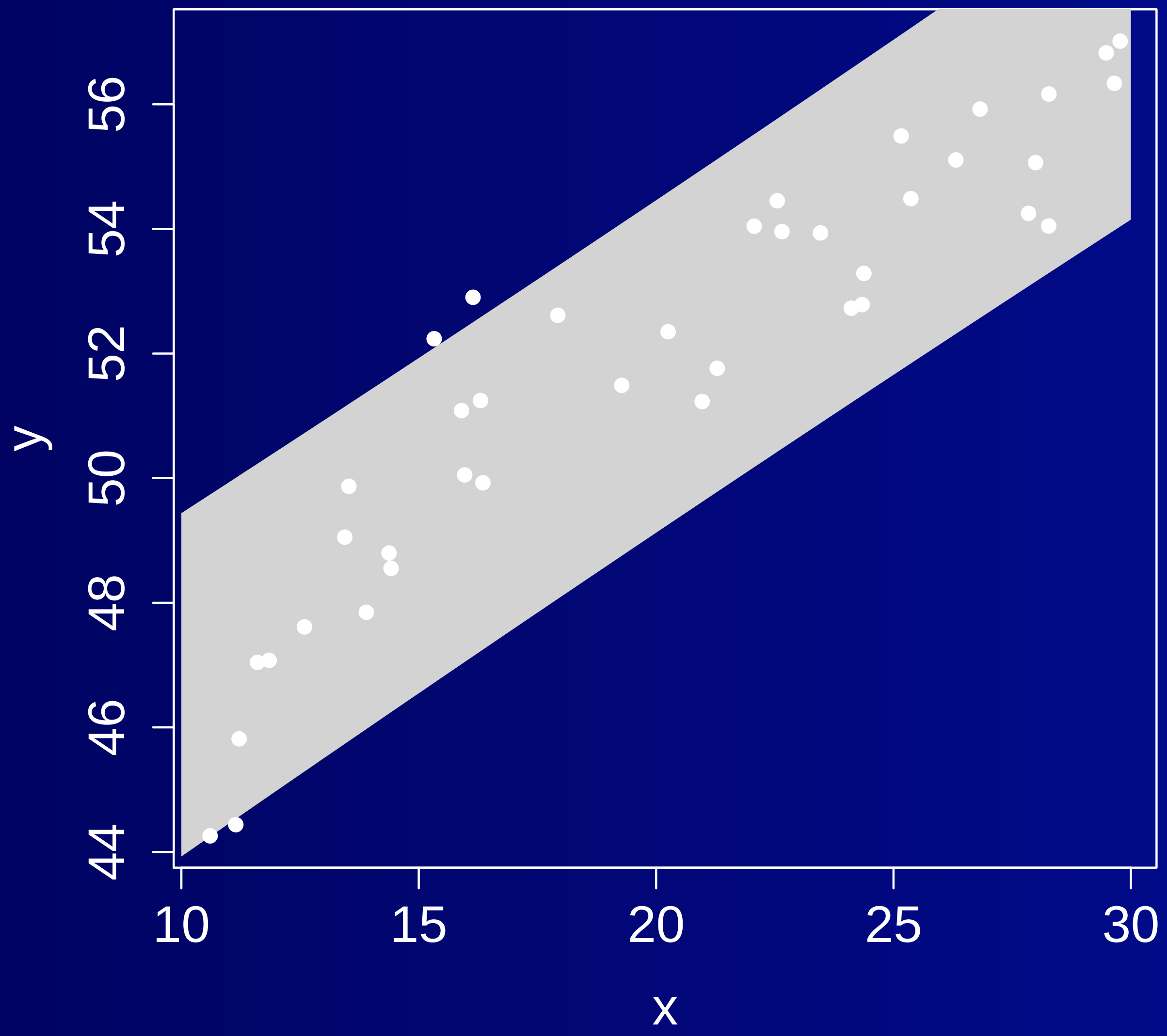
Linear regression



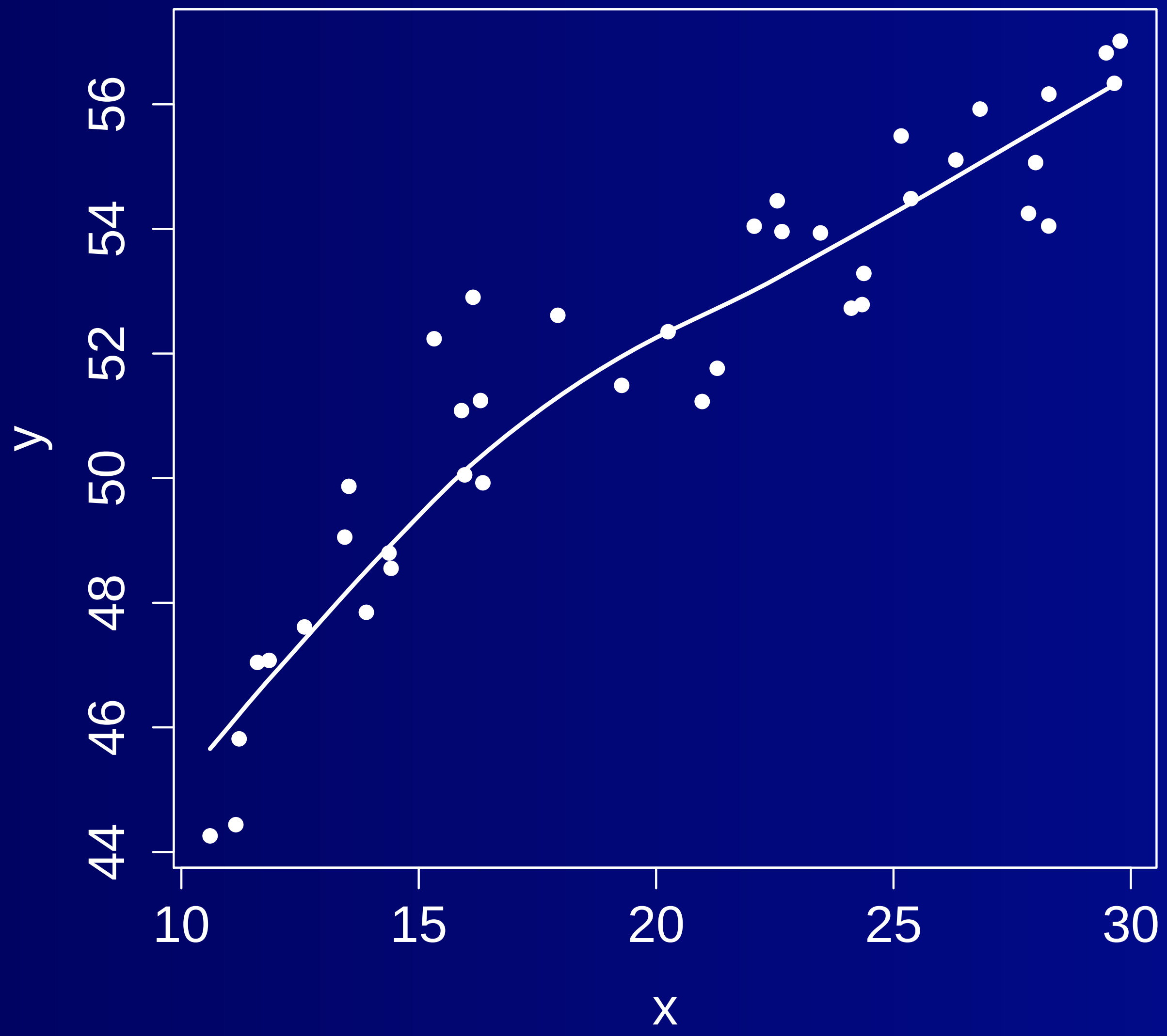
Linear regression: confidence interval



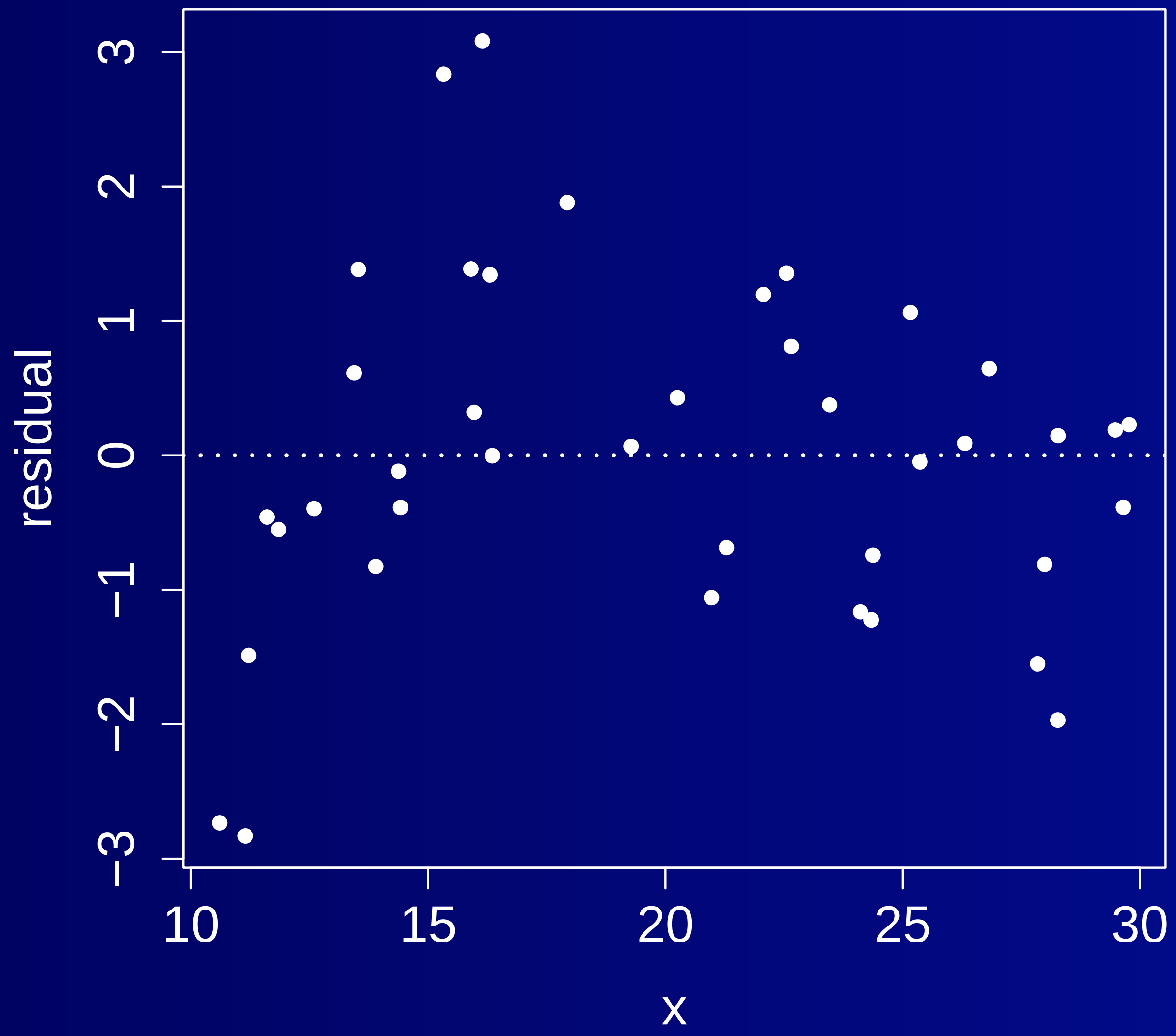
Linear regression: prediction interval



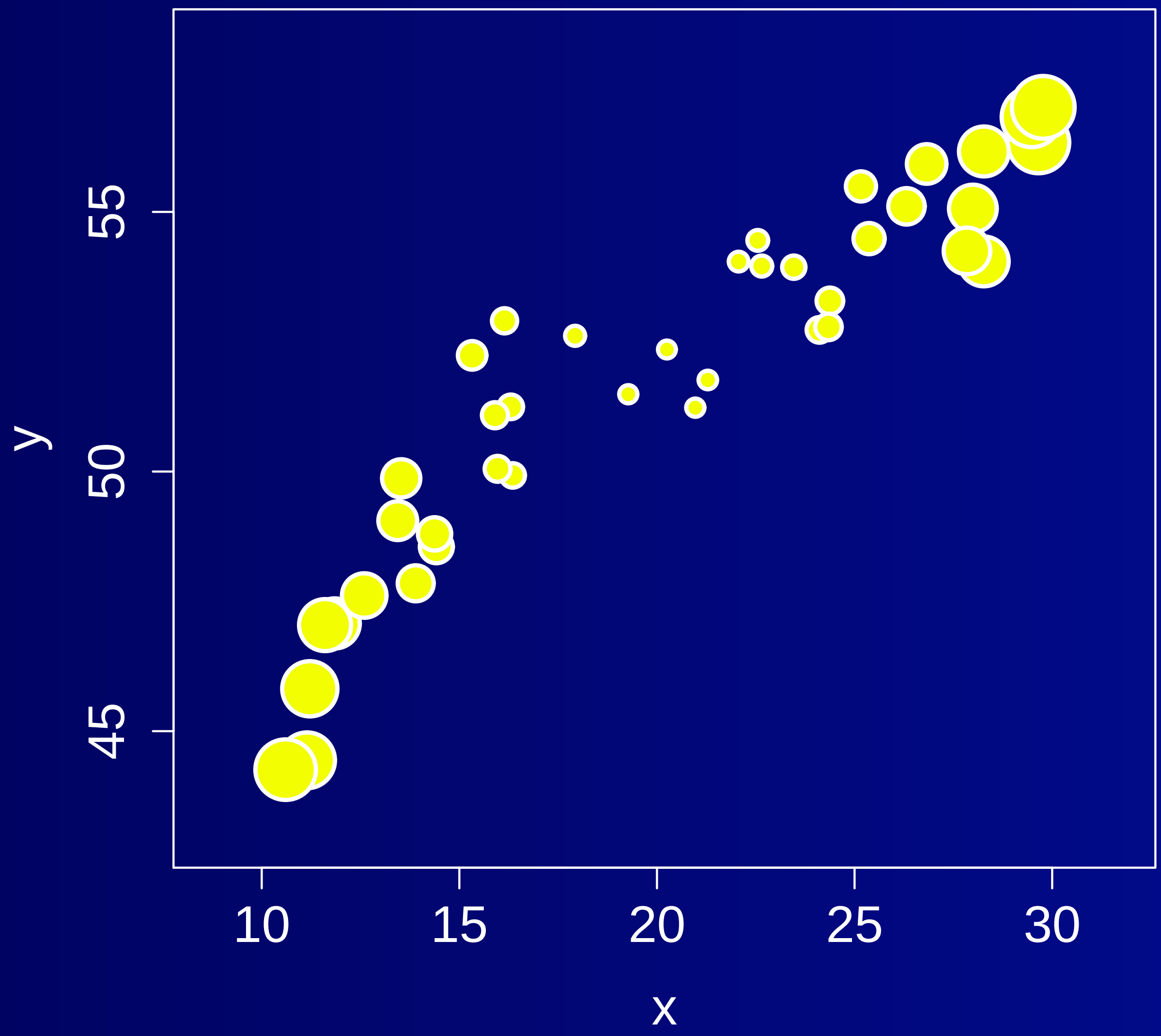
Nonparametric regression



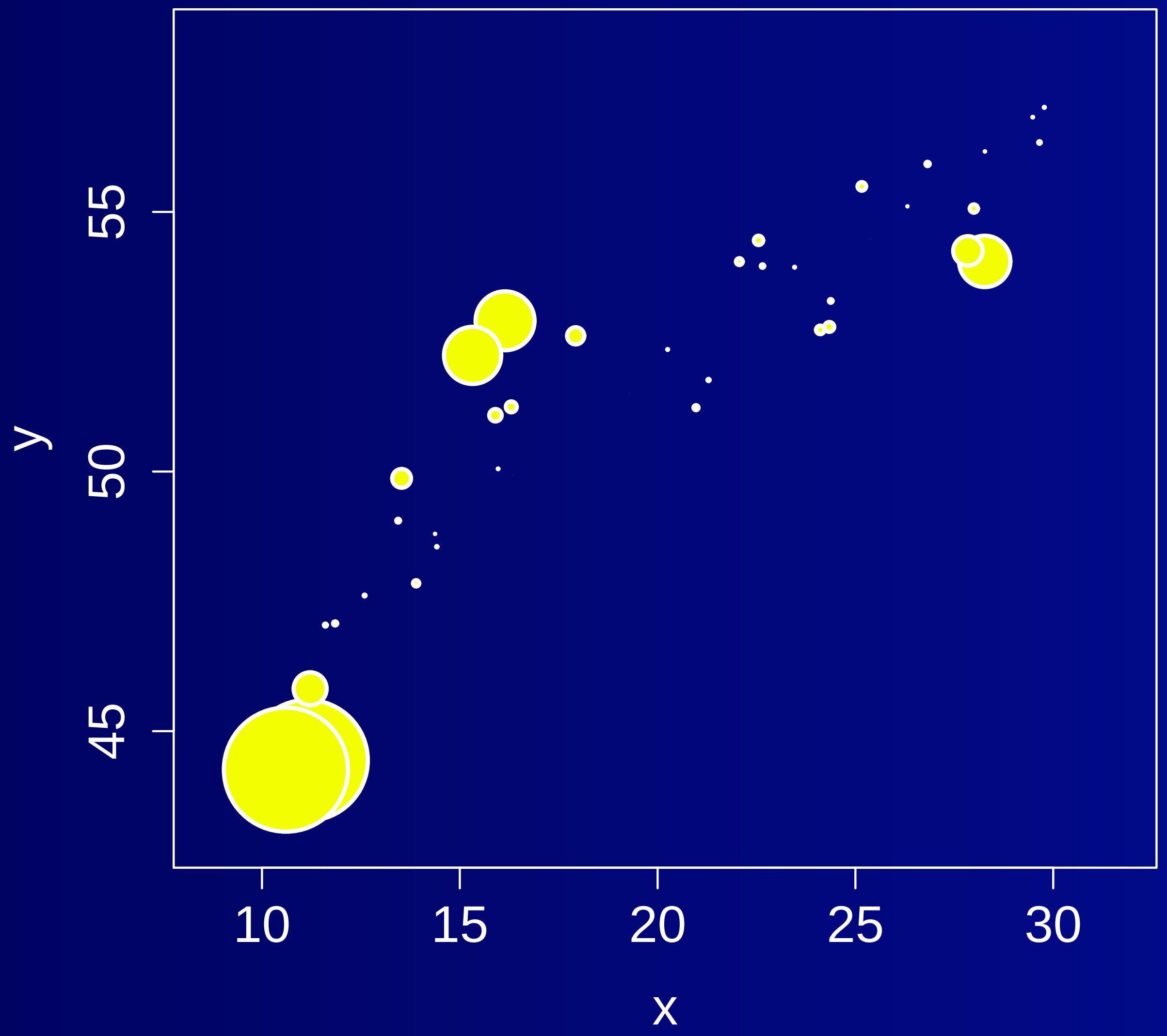
Linear regression diagnostics: residuals



Linear regression diagnostics: leverage

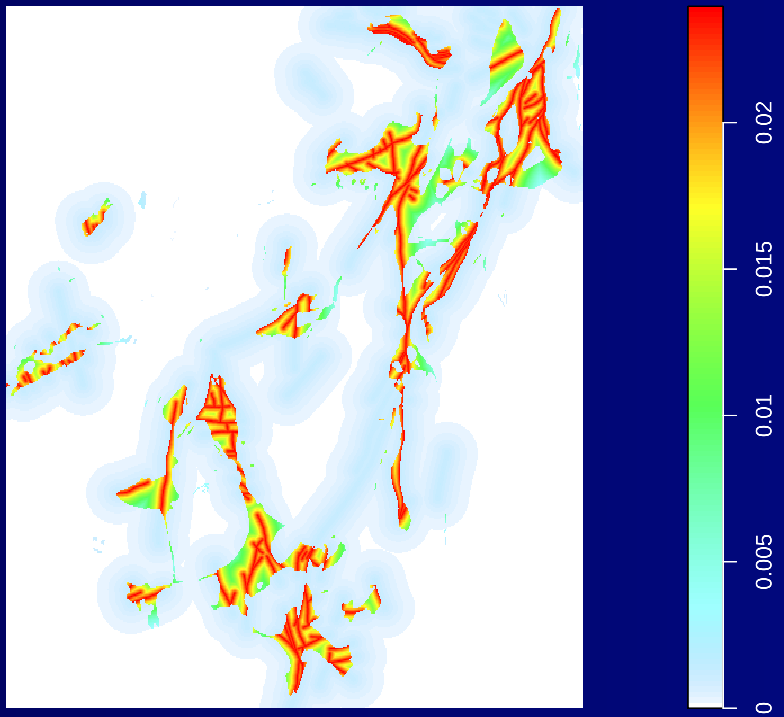


Linear regression diagnostics: influence



Gold deposits

Loglinear Poisson point process model



Predicted intensity of gold deposits
(number of deposits per km^2)

Assumes intensity is a loglinear function
of distance to nearest geological fault.

Validating the model

Logistic regression assumes

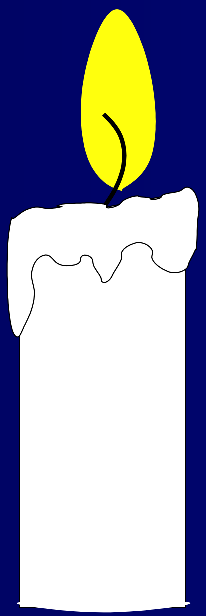
$$\lambda(u) = \exp(\beta_0 + \beta_1 X(u))$$

What if the relationship is not log-linear?

$$\lambda(u) = \rho(X(u))$$

How do we assess the evidence for/against a loglinear relationship?

Diagnostics for point process models



Extend Tukey's idea to diagnostics

1. write down a diagnostic for logistic regression
   (rescale appropriately for pixel size)

2. take very small pixels

3. interpret as a diagnostic for point processes

Using this bridge, *existing diagnostic tools from mainstream statistical science can be carried over to spatial point processes*

Diagnostics for point process models

1. residuals
2. leverage
3. influence
4. partial residual
5. nonparametric smooth

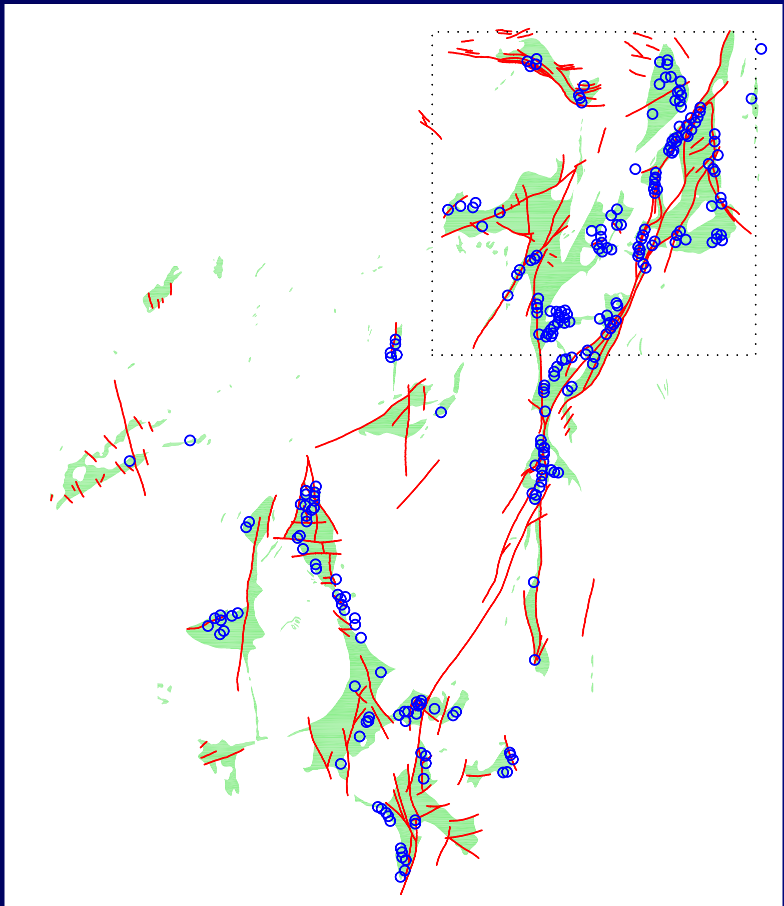
1. Residuals

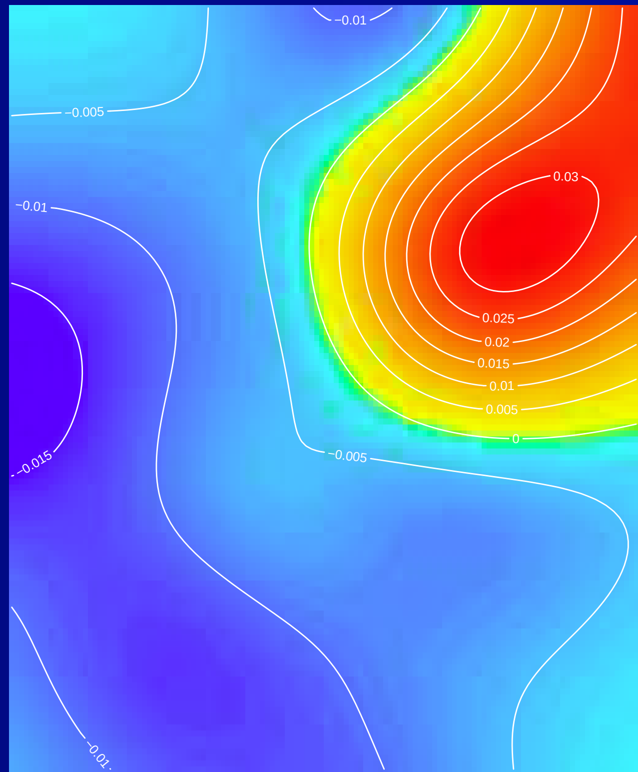
In linear regression, if $\hat{y}_i$ is the fitted mean for observation y_i, the residuals are

$$r_i = y_i - \hat{y}_i$$

The residuals should not show a systematic pattern. If they do, this suggests that the relationship between x and y has not been correctly modelled.

Gold deposits: smoothed Pearson residuals





Baddeley et al Residual analysis for spatial point processes. *J. Royal Statist. Soc. B* (2005)

Diagnostics for point process models

1. residuals
2. leverage
3. influence
4. partial residual
5. nonparametric smooth

2. Leverage

In linear regression of y on x,

- observed response: y_i
- fitted response: $\widehat{y_i}$
- *leverage*

$$h_i = \frac{\mathrm{d}\widehat{y_i}}{\mathrm{d}y_i}$$

measures how strongly the fitted value $\hat{y}_i$ depends on the observed value y_i.

Large values of leverage are associated with the observations which, because of their covariate value, have a potentially strong influence on the fitted model.

Leverage for Poisson point process

For a Poisson point process with loglinear intensity

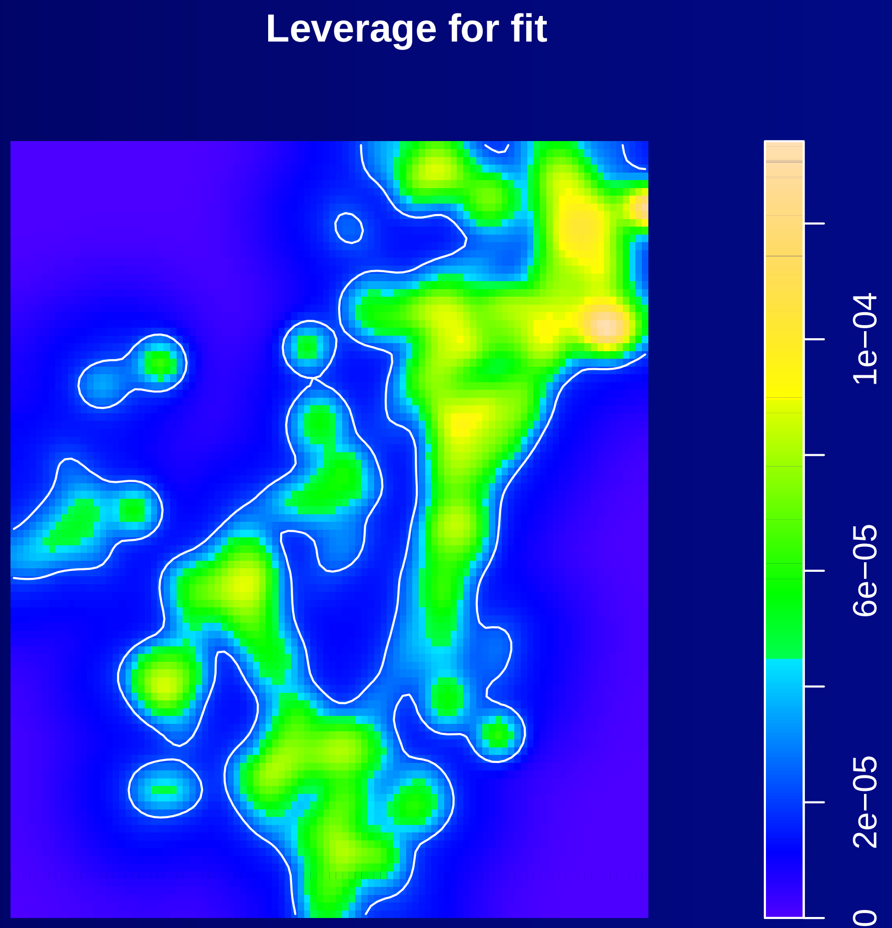
$$\lambda_{\boldsymbol{\beta}}(u) = \exp(\boldsymbol{\beta}^{\mathsf{T}} \boldsymbol{X}(u))$$

the leverage function is

$$h(u) = \lambda(u) \boldsymbol{X}(u) \mathcal{I}^{-1} \boldsymbol{X}(u)^{\mathsf{T}}$$

Baddeley, Chang & Song, Leverage and influence diagnostics for spatial point processes.
Scandinavian Journal of Statistics (2012)

Gold deposits: leverage



Diagnostics Poisson point process models

1. residuals
2. leverage
3. influence
4. partial residual
5. nonparametric smooth

3. Influence

In a linear model (etc), the *influence* of the ith observation is

$$s_i = \frac{2}{p} \log \frac{L(\hat{\boldsymbol{\theta}})}{L(\hat{\boldsymbol{\theta}}_{(-i)})}$$

where L is the likelihood, $\hat{\boldsymbol{\theta}}$ is the estimate of the parameter $\boldsymbol{\theta}$ using all the data, $\hat{\boldsymbol{\theta}}_{(-i)}$ is the estimate using all the data except the ith observation, and p is the number of parameters.

Large values of influence are associated with the observations which, because of their atypical response *and* high leverage, actually had a strong effect on the fitted model.

Influence for loglinear Poisson model

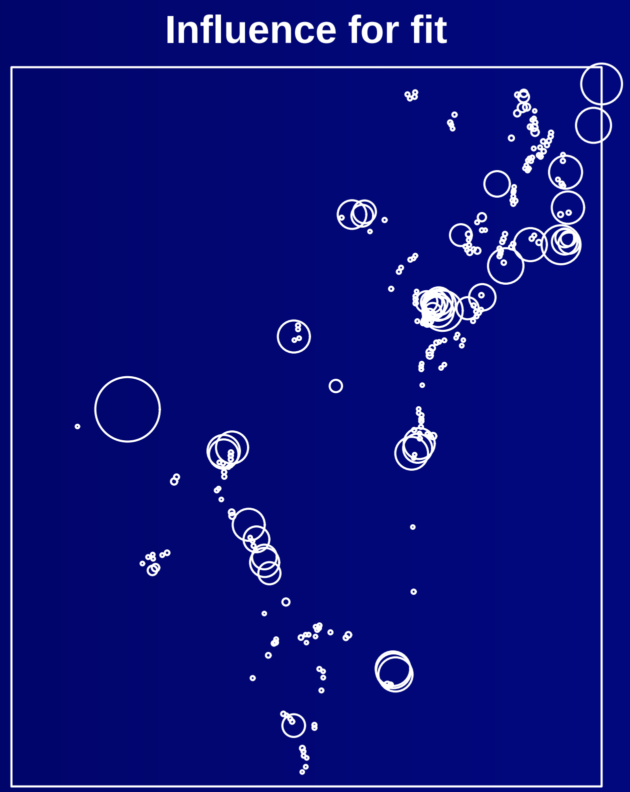
For the loglinear Poisson point process, the influence of data point s_i is

$$m_i = \frac{1}{p} \boldsymbol{X}(s_i) \mathcal{I}_{\hat{\boldsymbol{\beta}}}^{-1} \boldsymbol{X}(s_i)^\top .$$

Baddeley, Chang & Song, Leverage and influence diagnostics for spatial point processes.

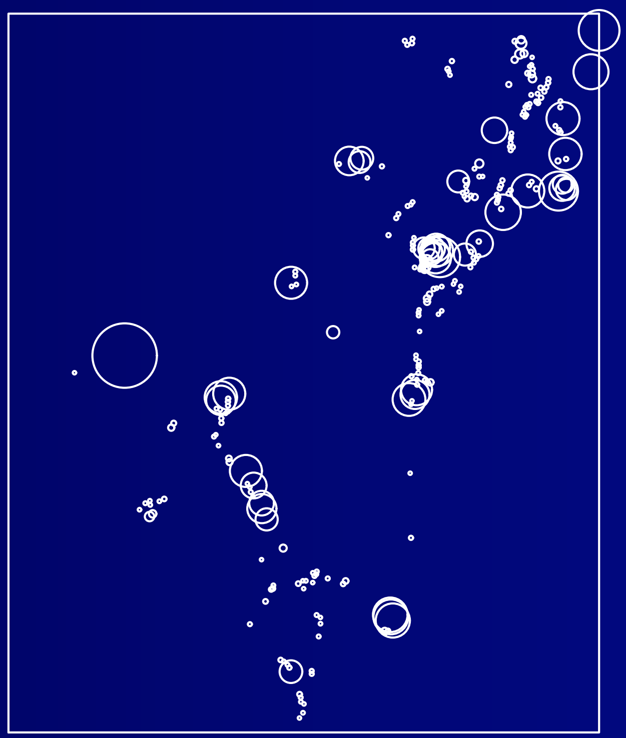
Scandinavian Journal of Statistics (2012)

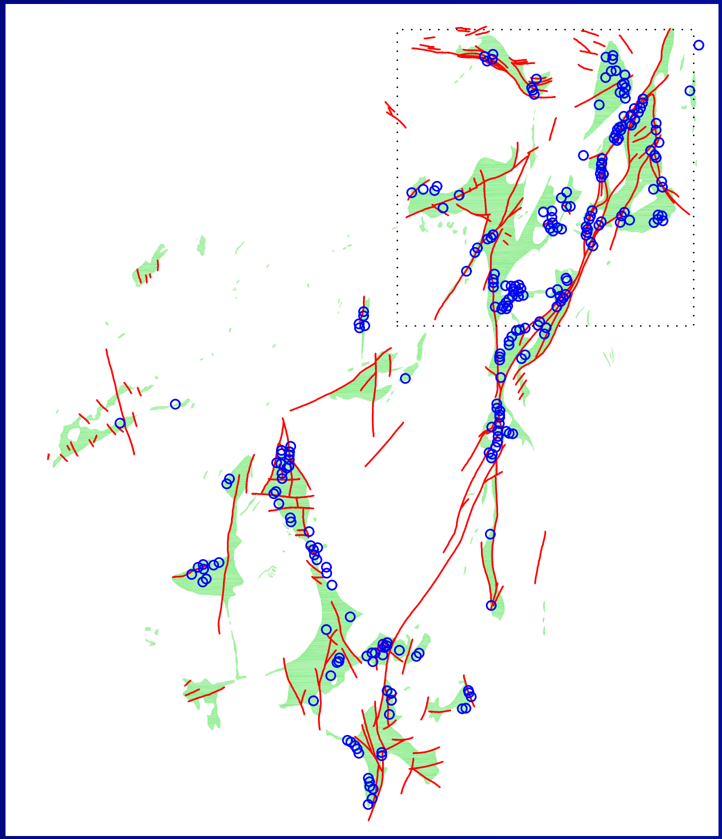
Gold deposits: influence



Gold deposits: influence

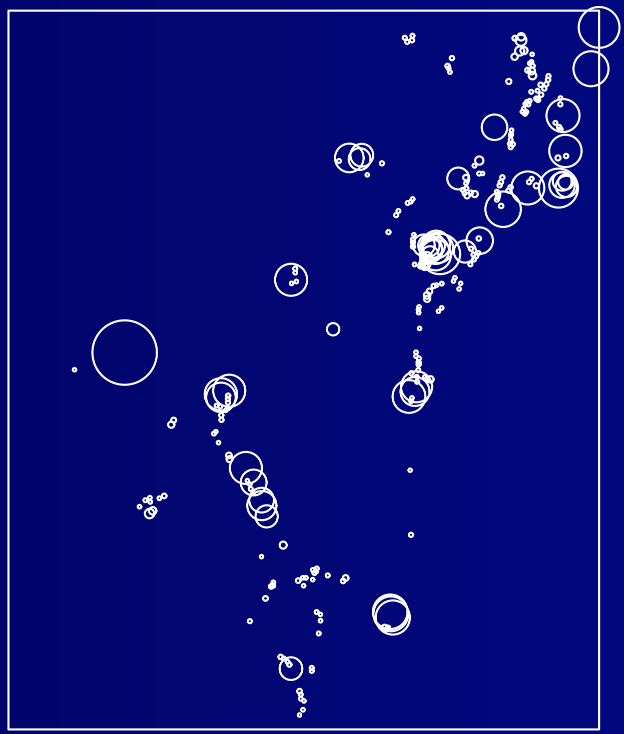
Influence for fit





Gold deposits: influence

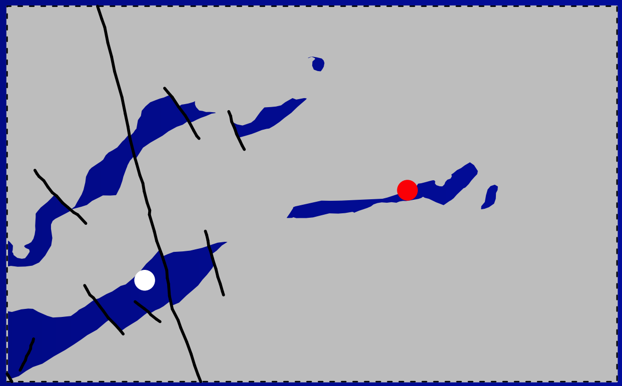
Influence for fit



Large circle at left

identifies an outlier

or anomaly



Diagnostics for point process models

1. residuals
2. leverage
3. influence
4. partial residual
5. nonparametric smooth

4. Partial residuals

In linear regression

$$y = ax + b$$

the partial residual (aka component-plus-residual) is

$$r_i = \hat{b}\, x_i + \frac{y_i - \hat{y}_i}{\hat{\sigma}^2}.$$

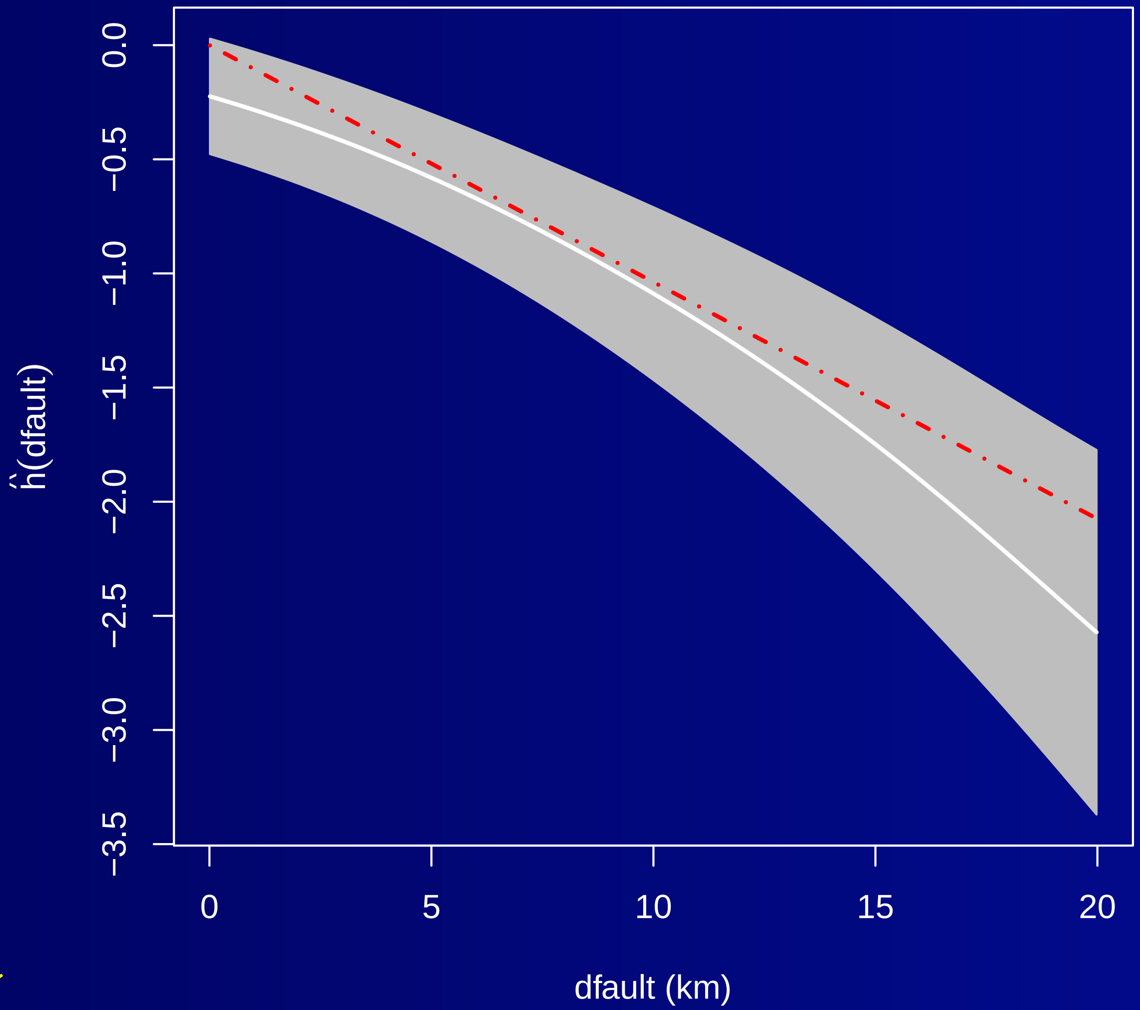
A smoothed plot of r_i against x_i gives an estimate of the true relationship between x and y.

Partial residuals for spatial point process model

For loglinear Poisson point process, the partial residuals are the values of $\boldsymbol{X}$ at the data points s_i with weights $1/\lambda(s_i)$.

Baddeley, Chang, Song & Turner, Residual diagnostics for covariate effects in spatial point process models. *J. Computational and Graphical Statistics* (2012)

Gold deposits: partial residual plot



Diagnostics for point process models

1. residuals
2. leverage
3. influence
4. partial residual
5. nonparametric smooth

5. Nonparametric estimate of covariate effect

Suppose that, instead of the loglinear model, the point process intensity depends on covariate $\boldsymbol{X}$ through

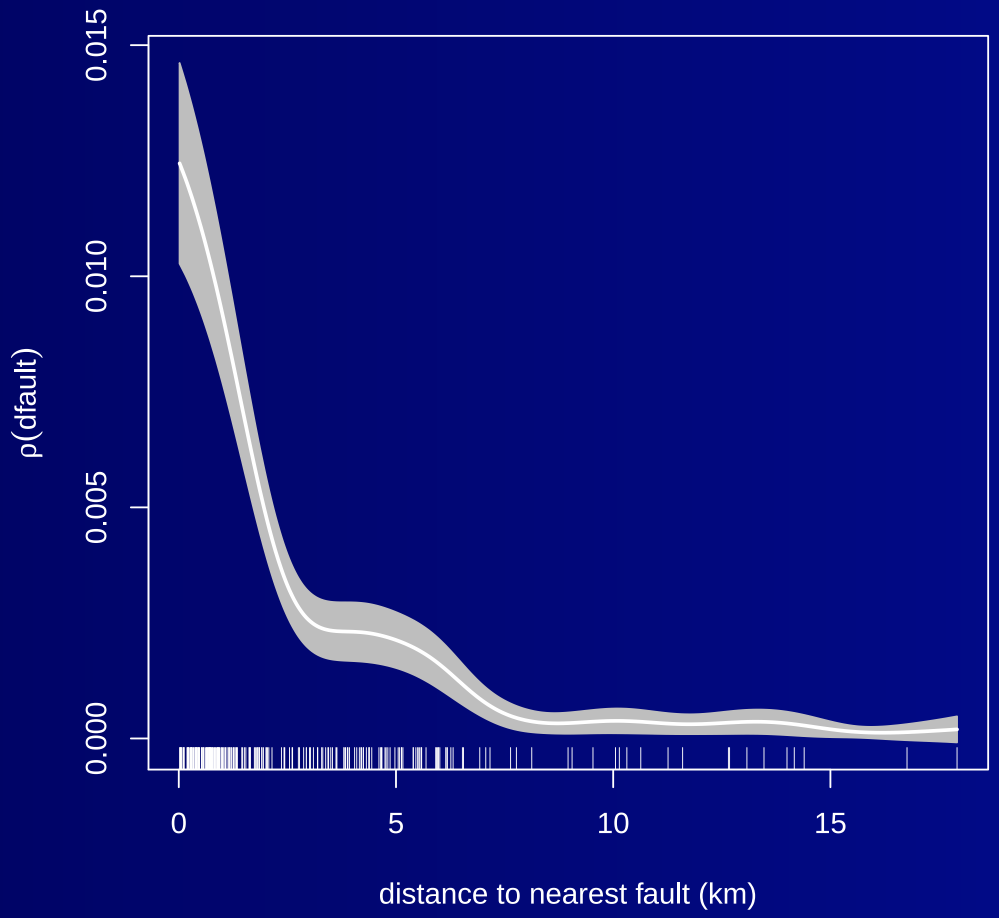
$$\lambda(u) = \rho(\boldsymbol{X}(u))$$

where the function ρ is to be estimated.

Estimate ρ by a kernel smoothing technique

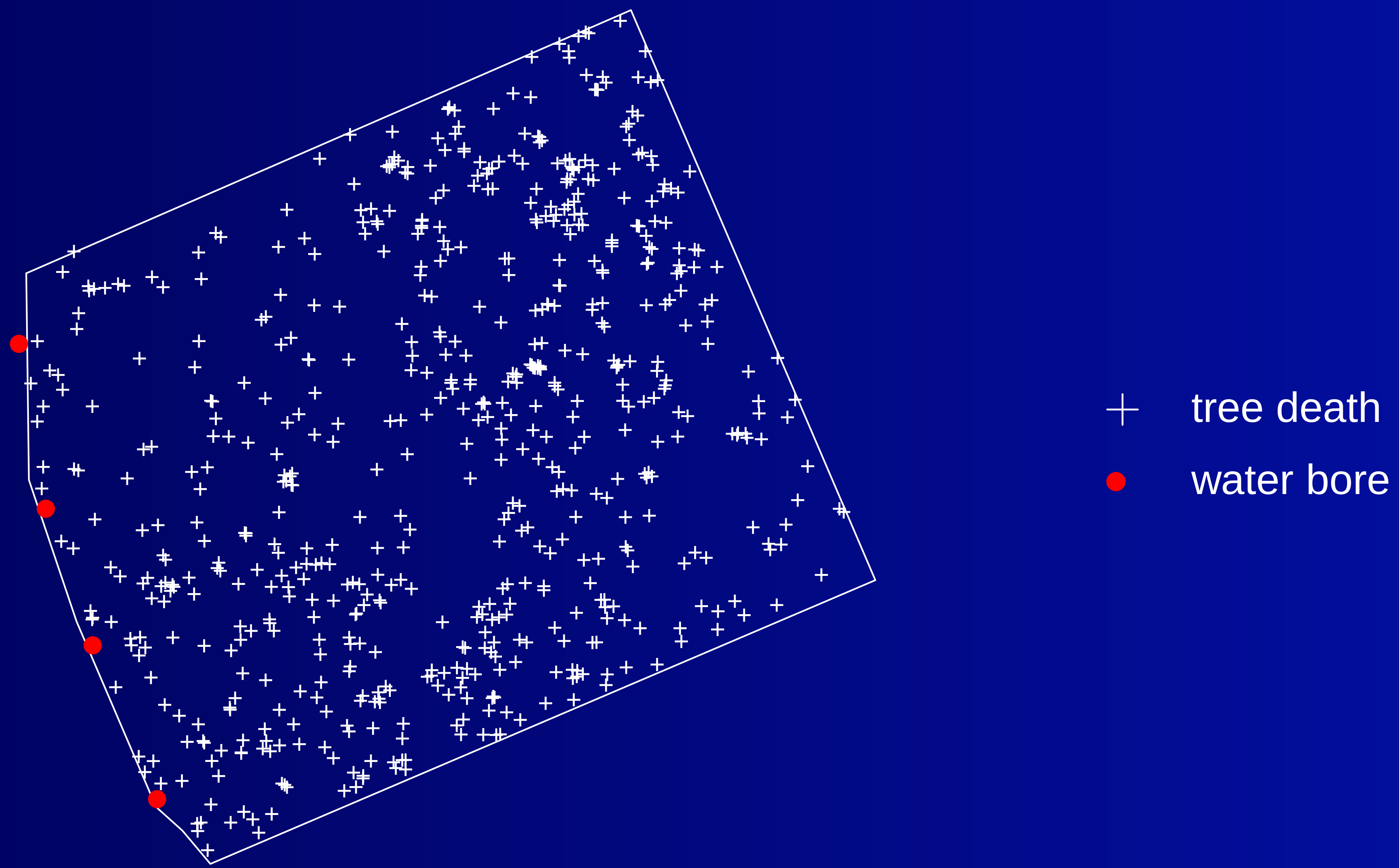
Baddeley, Chang, Song & Turner, Nonparametric estimation of the dependence of a spatial point process on a spatial covariate. *Statistics and its Interface* **5** (2012)

Gold deposits: smoothed effect estimate

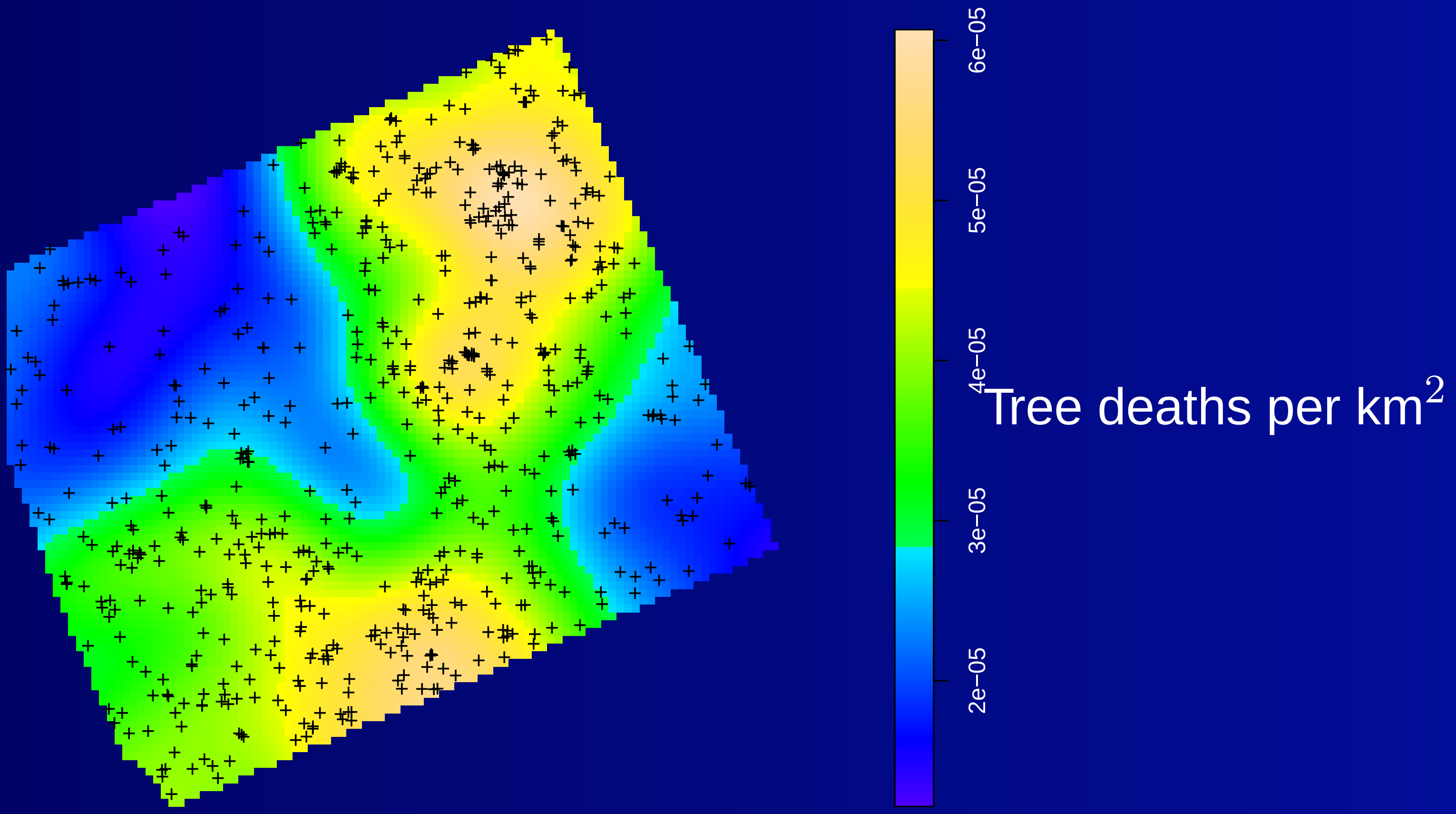


new

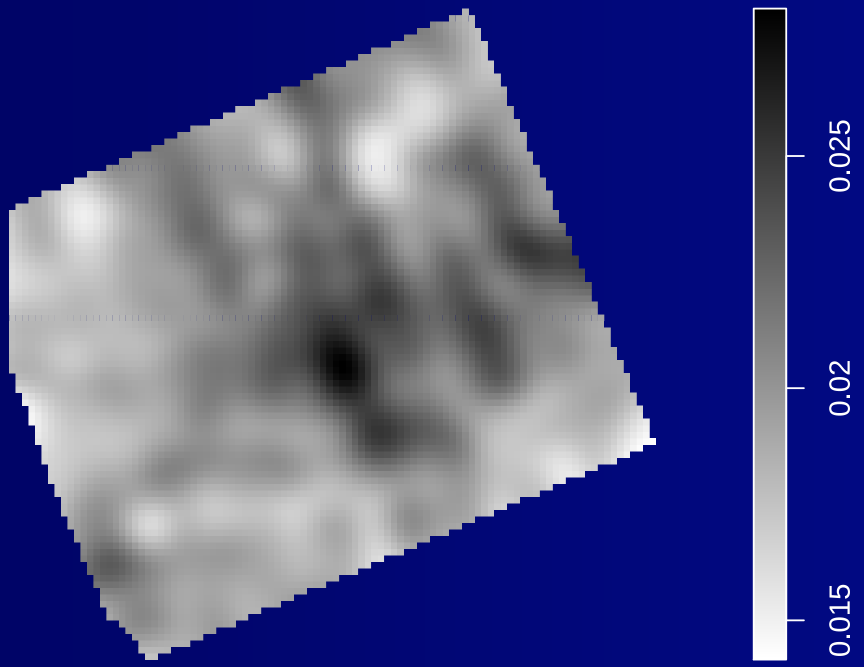
Tree deaths in Perth’s groundwater catchment



Spatially varying death rate



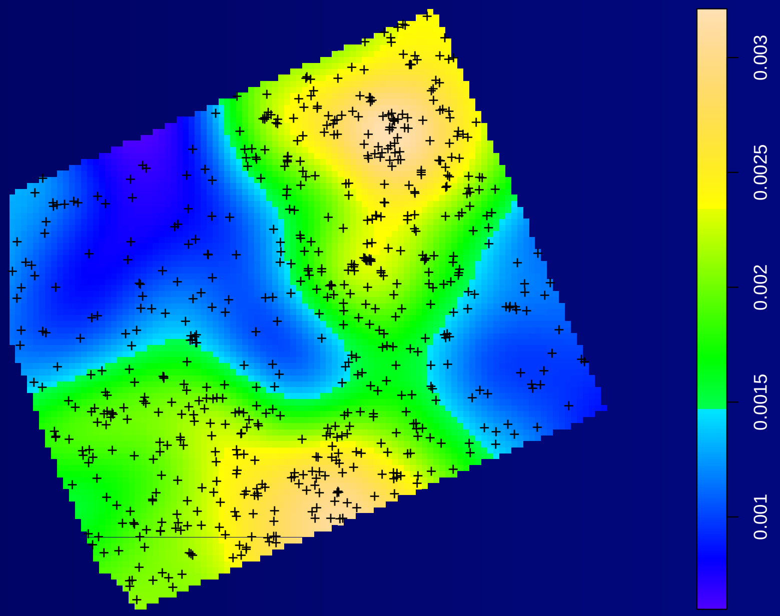
Density of live trees



Compiled from 300,000 tree locations (detected from aerial imagery)

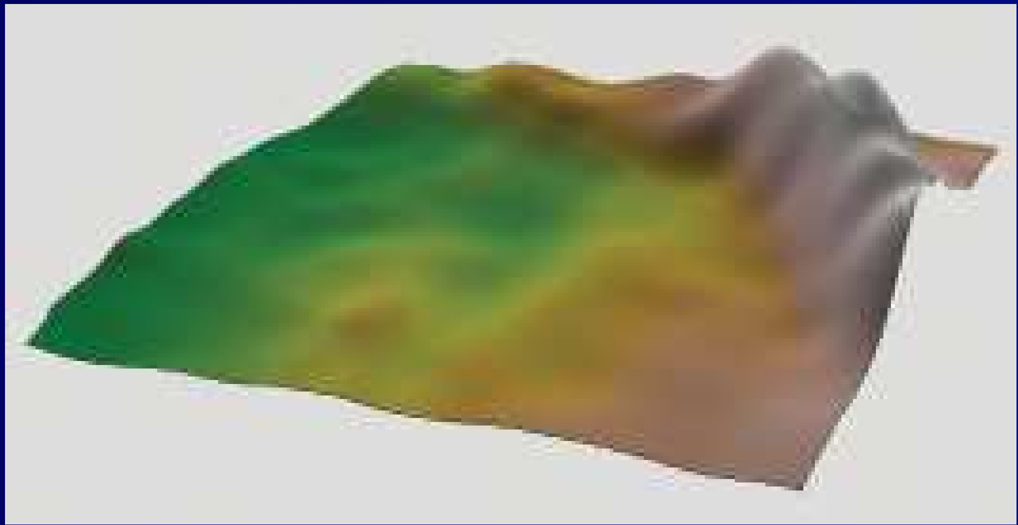
Y.M. Chang

Spatially varying death risk

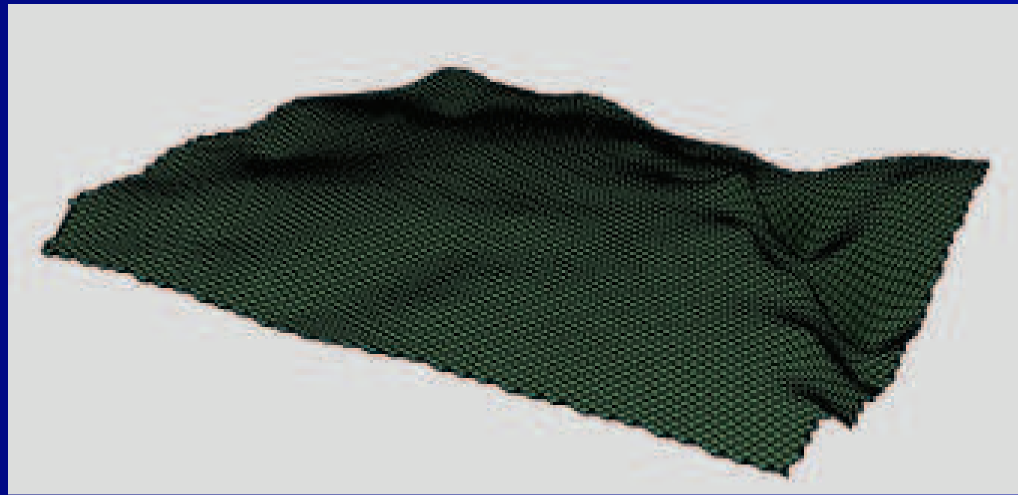


Deaths per thousand trees

Covariate data



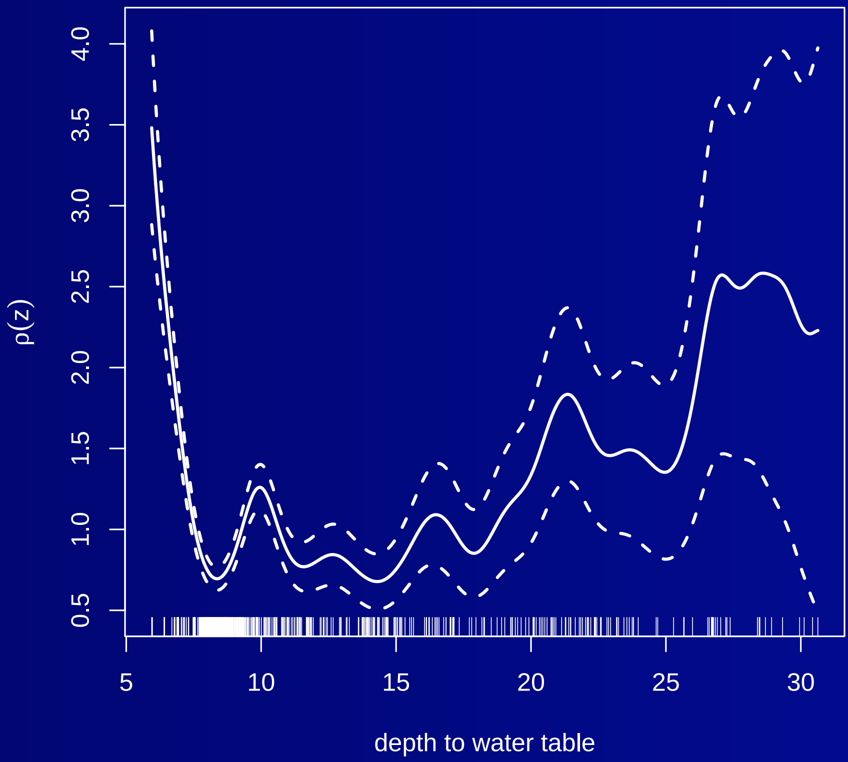
Terrain elevation



Depth to water table

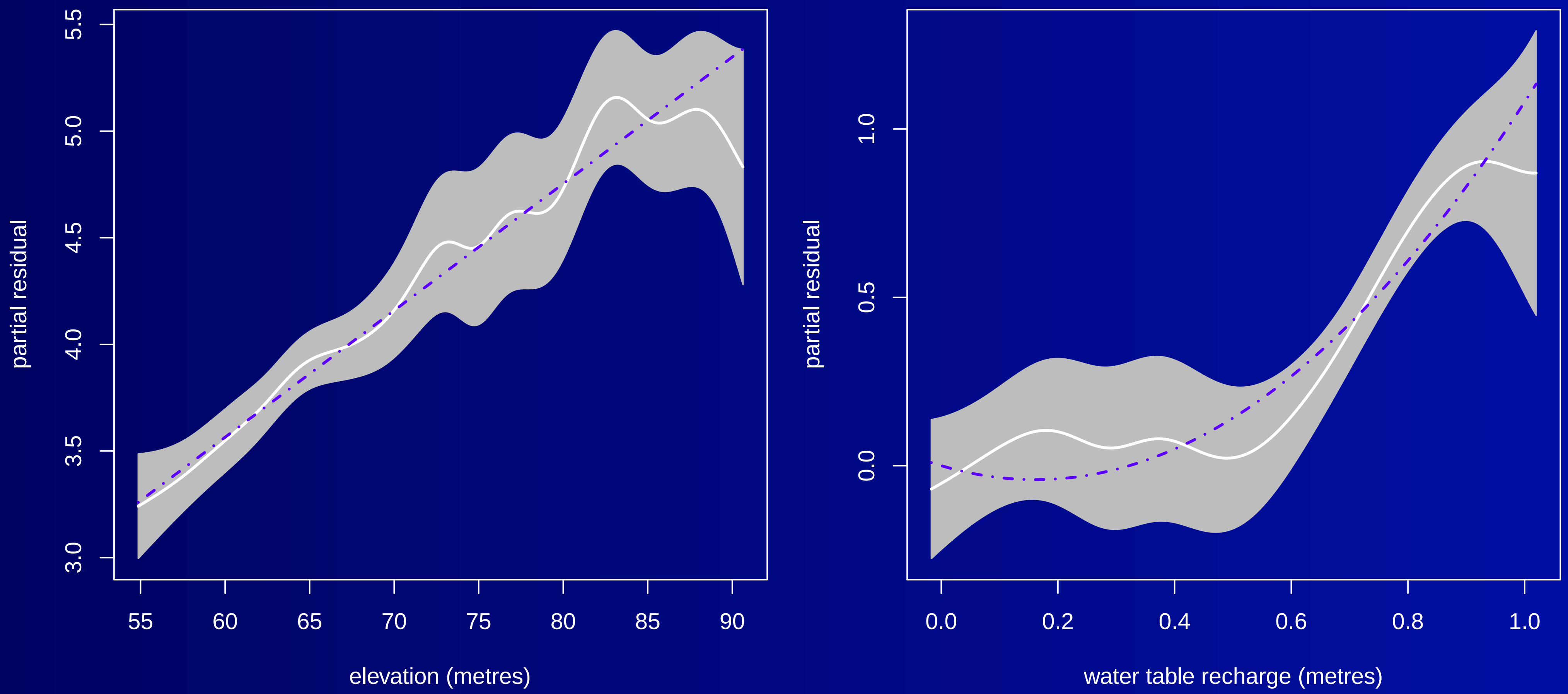
Effect of depth to water table

Nonparametric estimate



Effect of terrain elevation, groundwater recharge

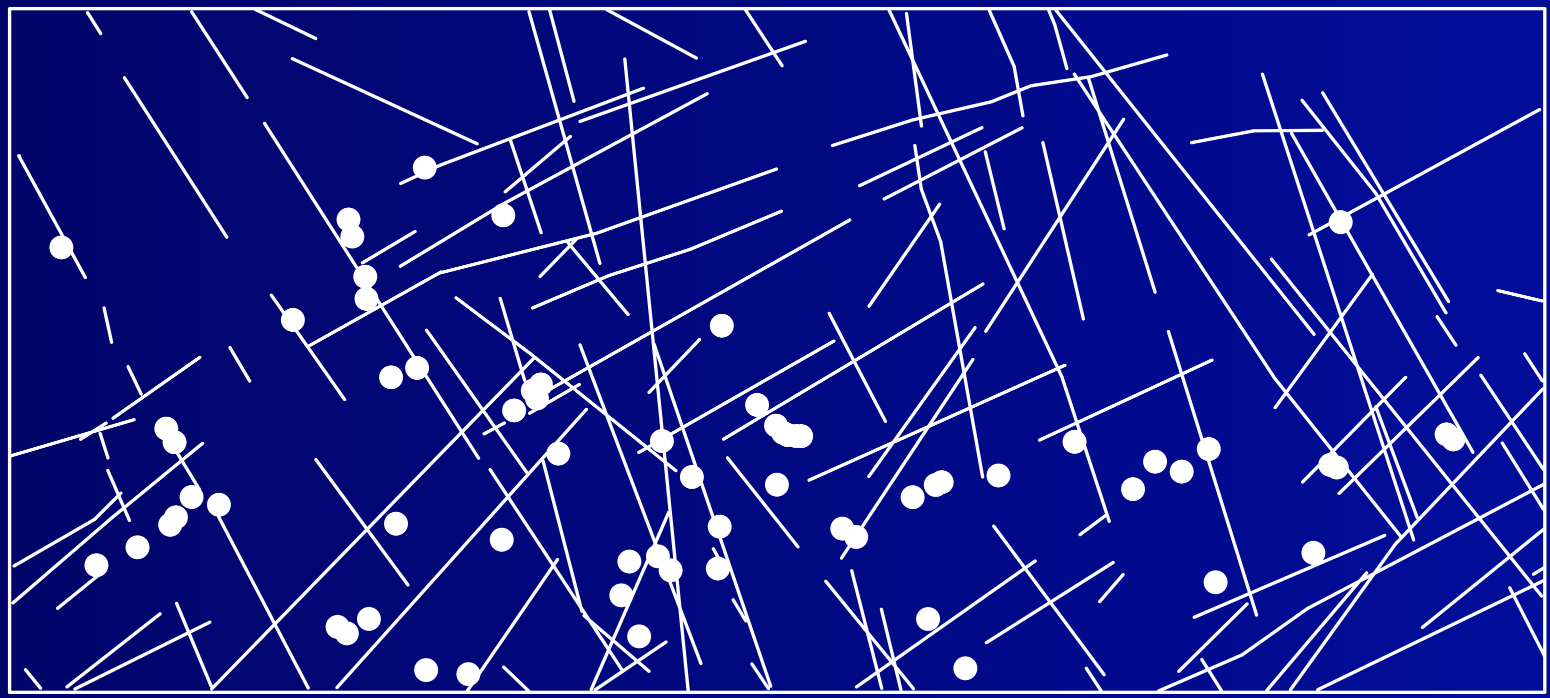
Partial residuals



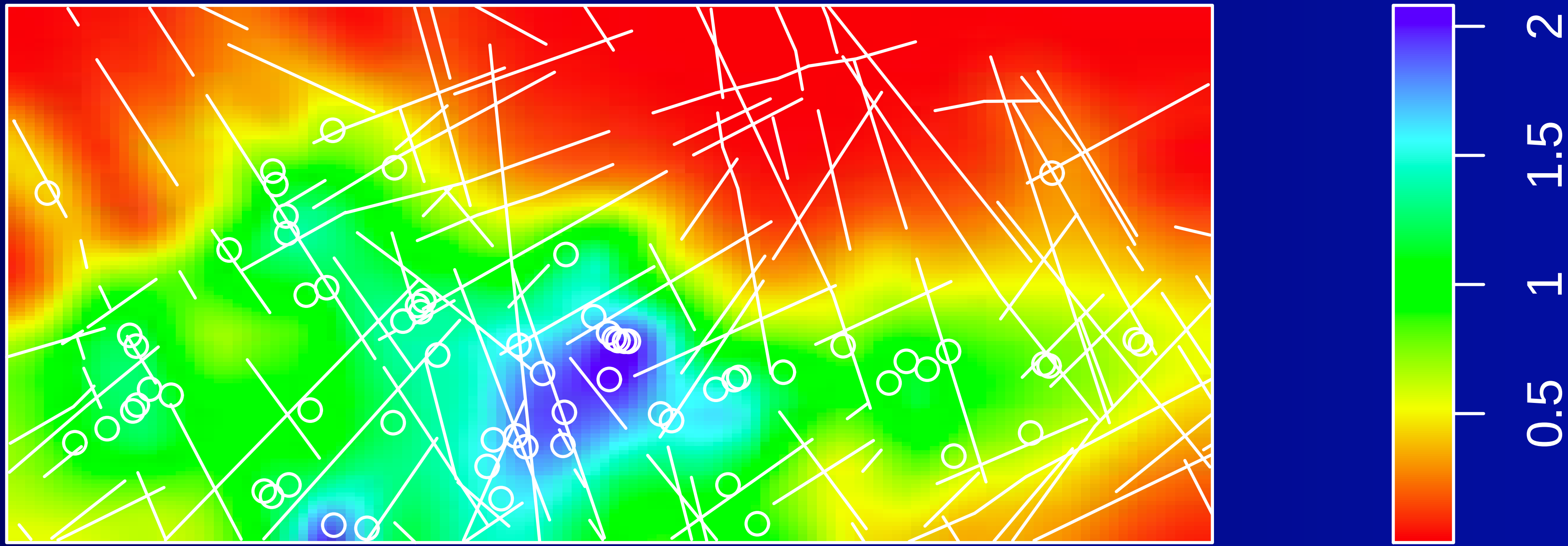
Chang et al, Spatial statistical analysis of tree deaths using airborne digital imagery, *Intl. J. Appl. Earth Observ. & Geoinformation* (2012)

Coming Soon . . .

Local Likelihood

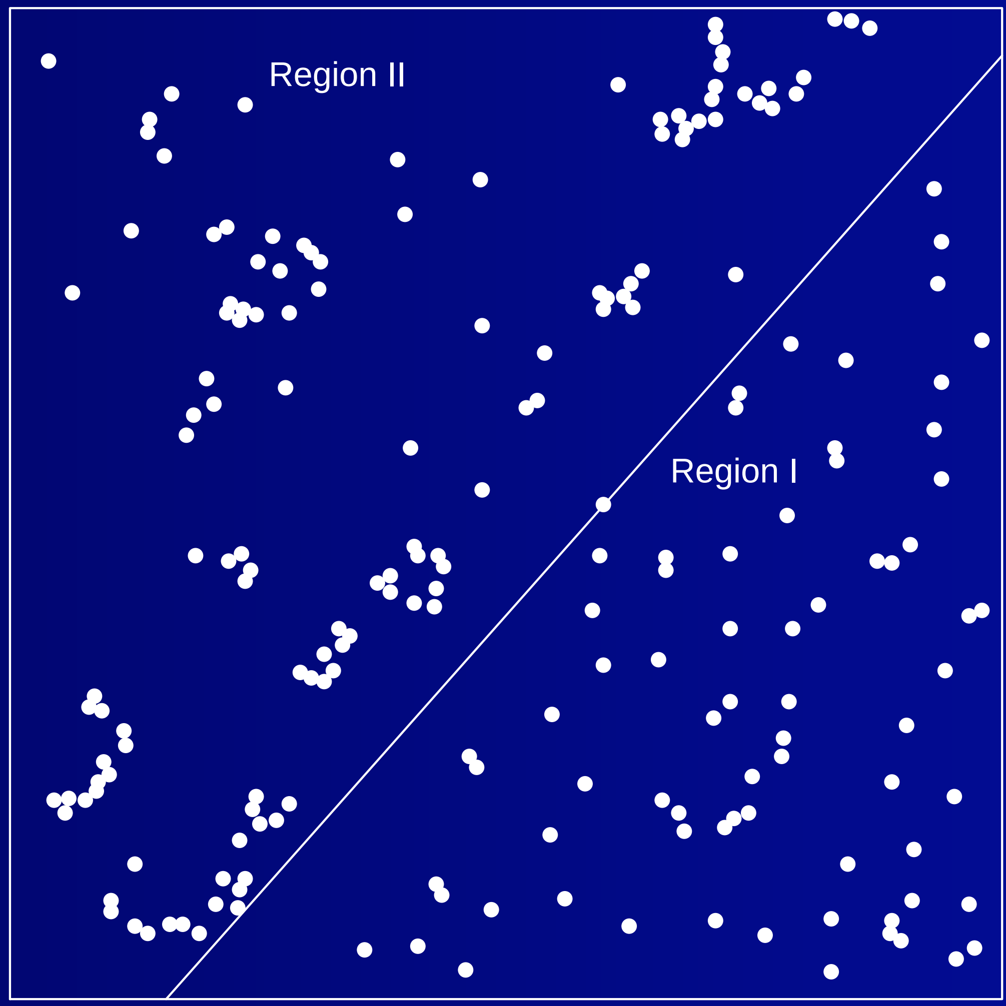


Copper deposits and lineaments, Queensland



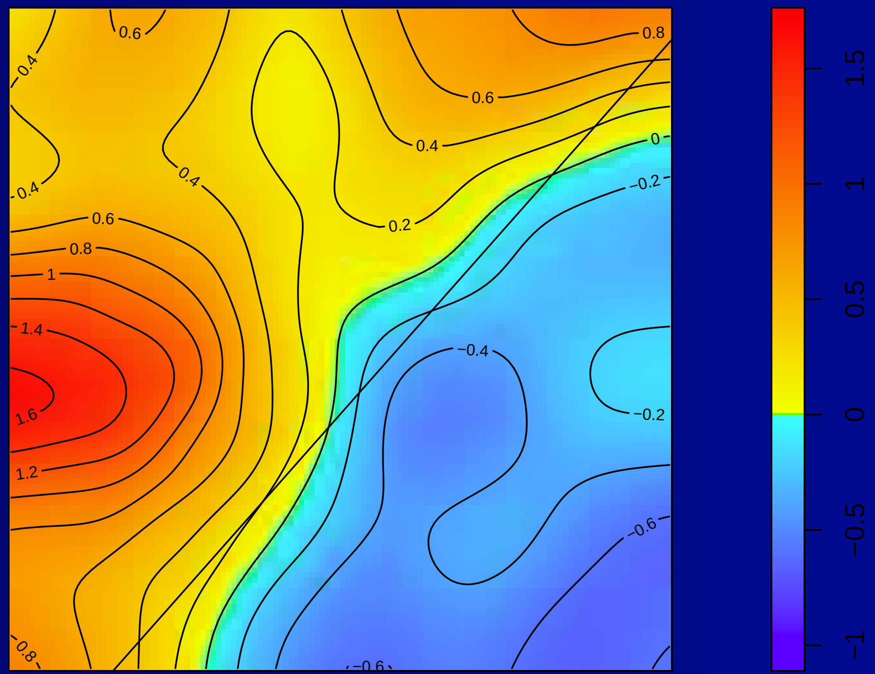
Local Poisson loglinear model

Local composite likelihood



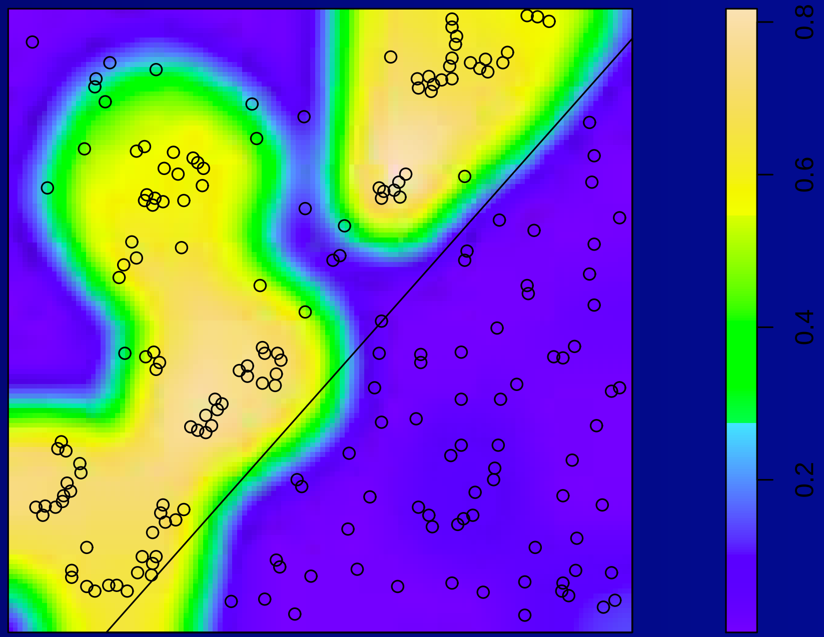
California Redwood saplings

Local composite likelihood



Local Gibbs point process model

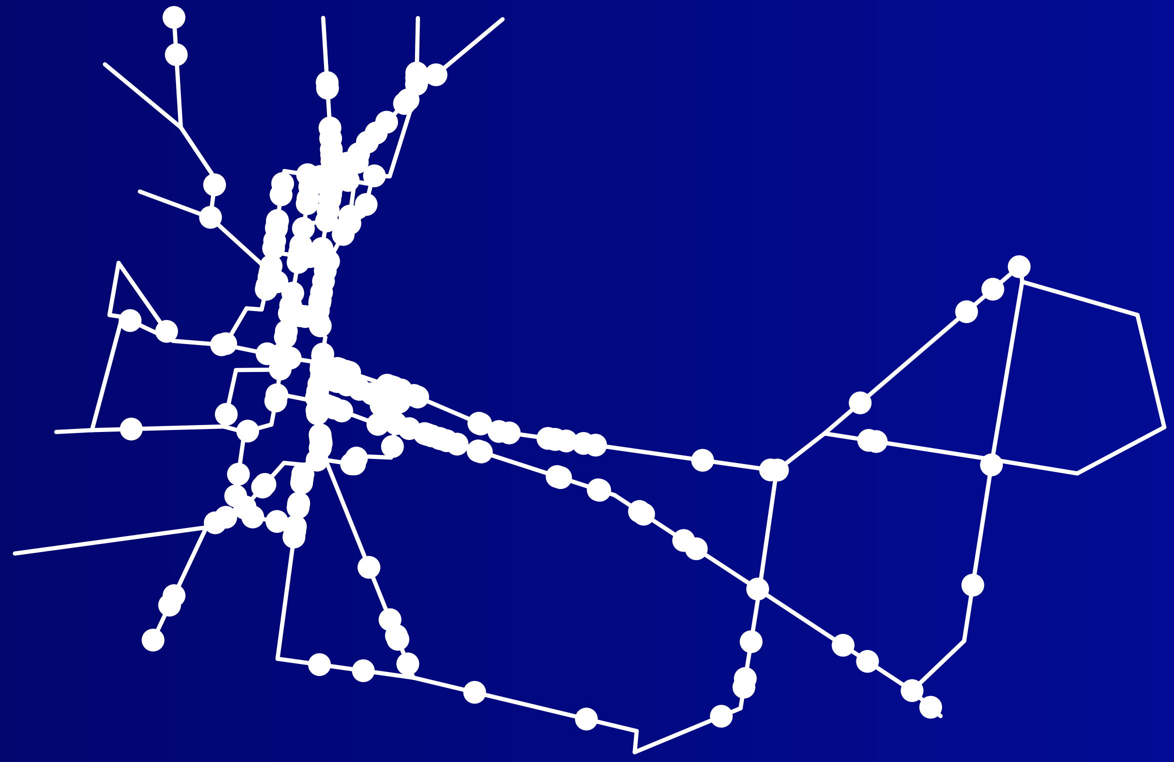
Local composite likelihood



Local Neyman-Scott cluster process model

Point patterns on linear networks

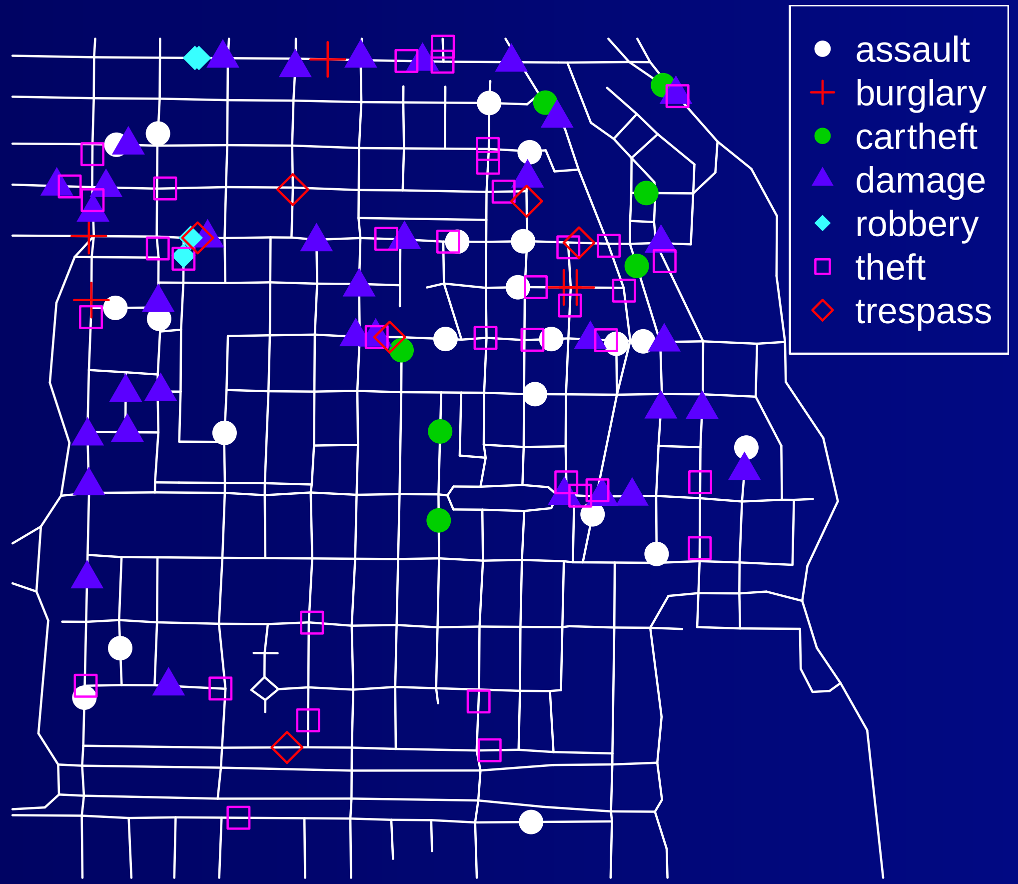
Road accidents in Geelong 2010–2012



G. McSwiggan

Point patterns on linear networks

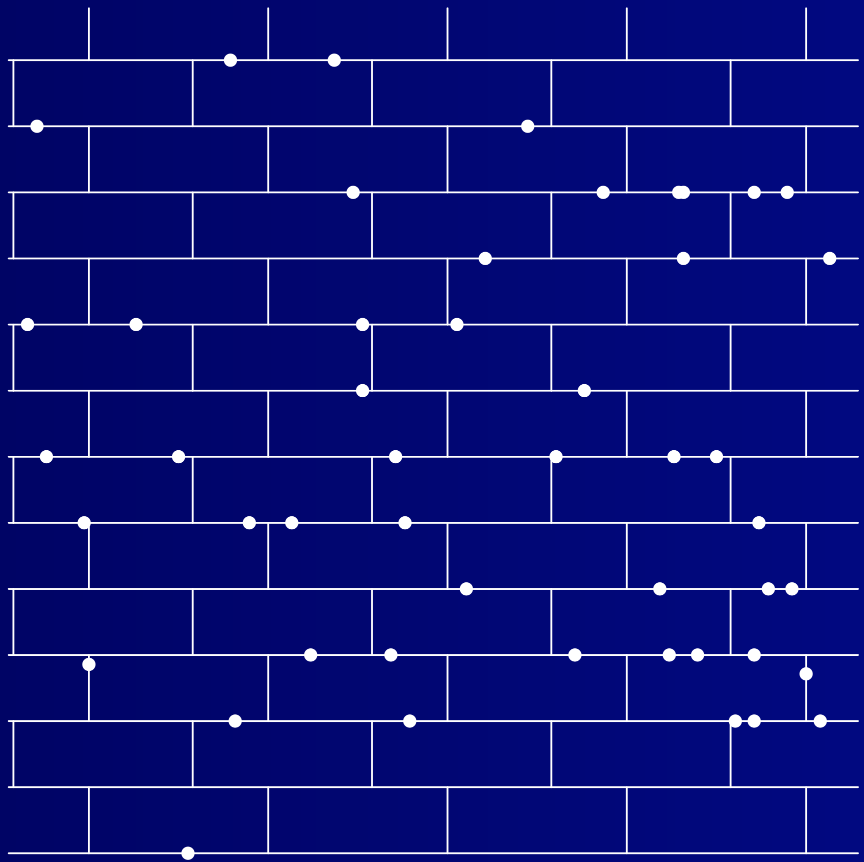
Chicago street crimes



Chicago Weekly News

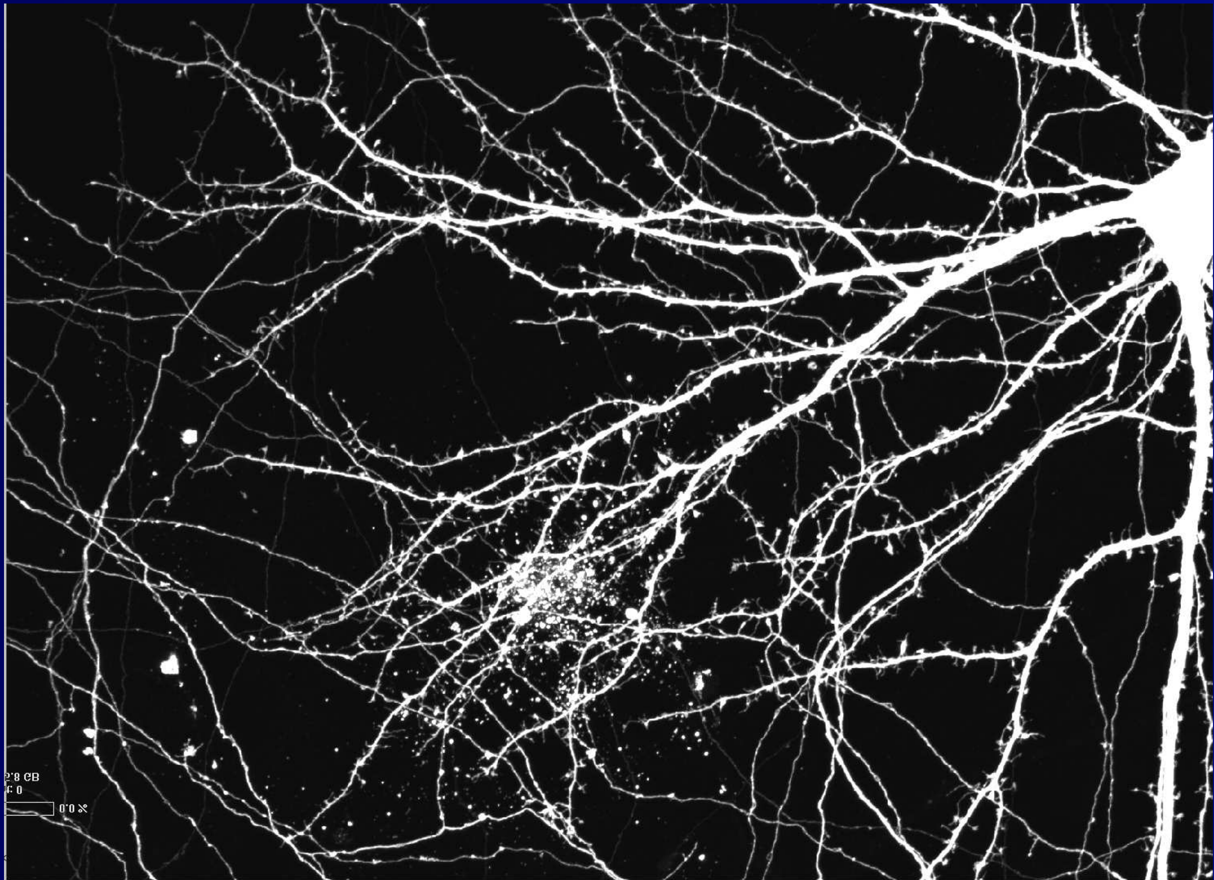
Point patterns on linear networks

Spider webs on a brick wall



Point patterns on linear networks

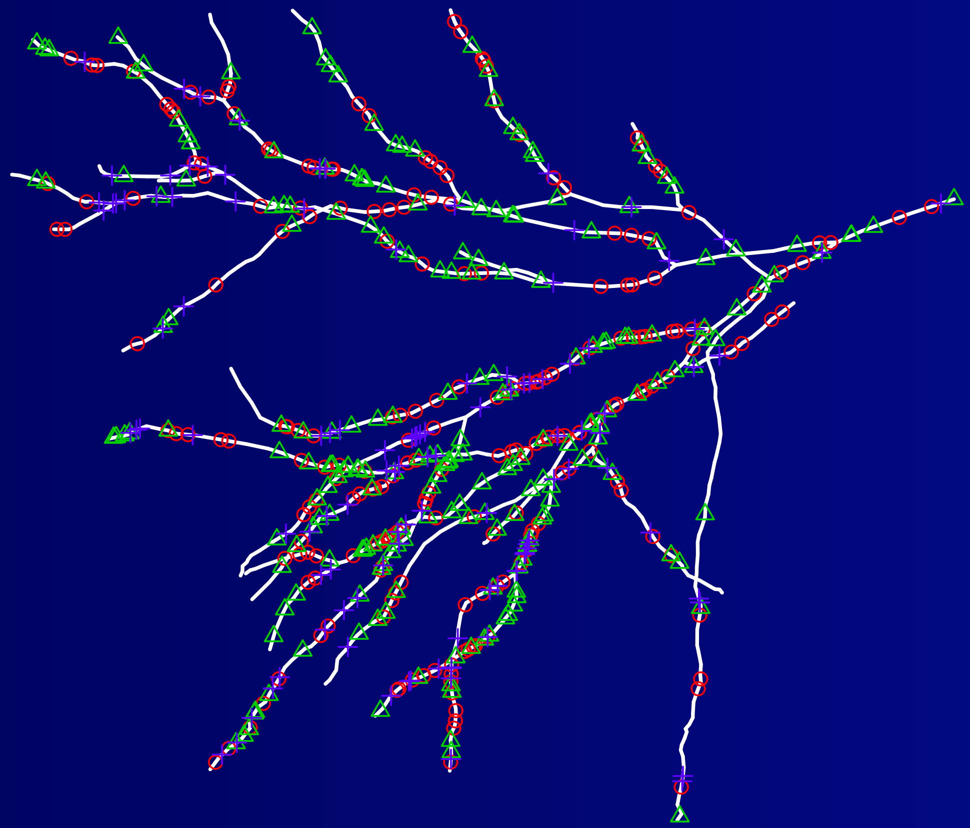
Dendritic spines



Kosic Lab, UCSB

Point patterns on linear networks

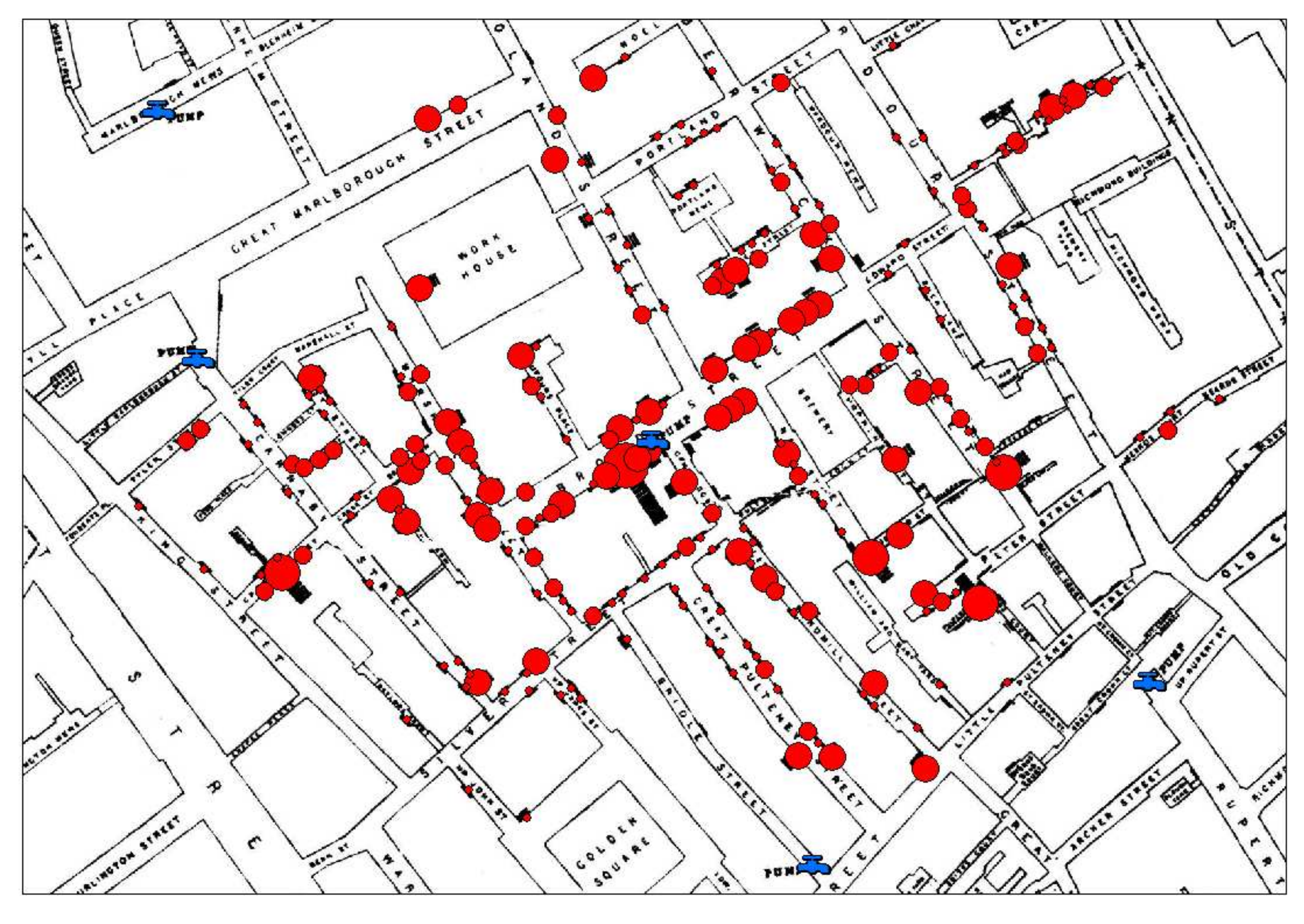
Dendritic spines



Kosic Lab, UCSB

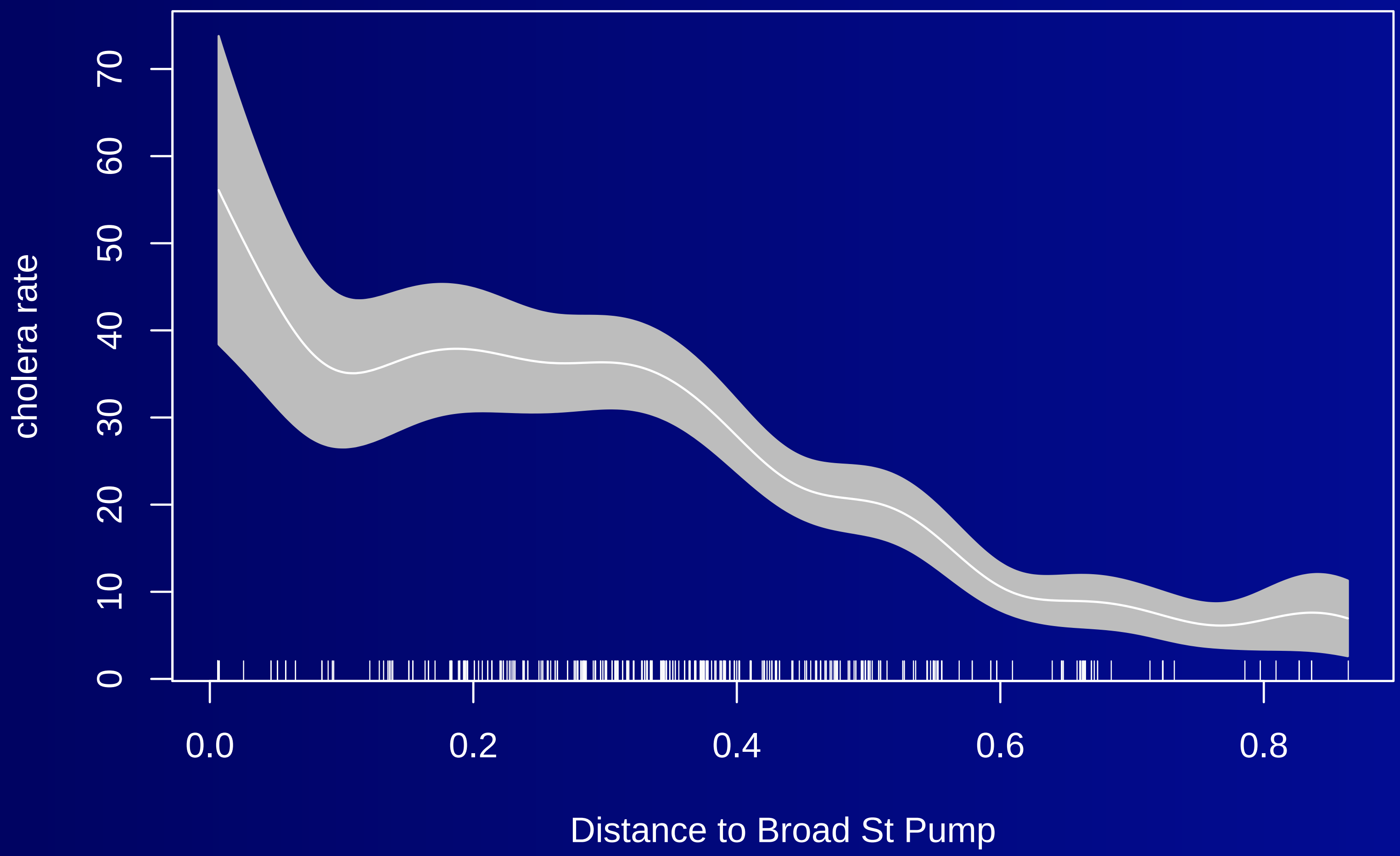
A. Jammalamadaka

Cholera in Soho 1854



Cholera in Soho 1854

Cholera in Soho 1854



Cholera in Soho 1854

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