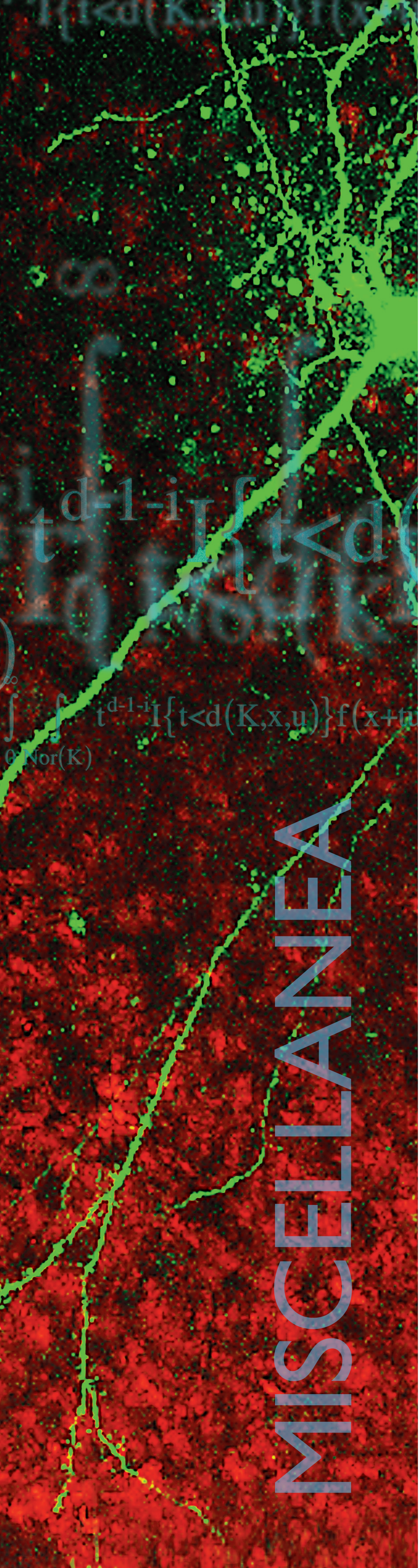

CENTRE FOR **STOCHASTIC GEOMETRY**
AND ADVANCED **BIOIMAGING**

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Erasmus Lectures on Rotational integral geometry

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Rotational integral geometry

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These lecture notes contain an introduction to rotational integral geometry. A short presentation of recent results for rotational integral geometry of tensor valuations is also given and open problems in rotational integral geometry are discussed.

The motivation for developing rotational integral geometry comes from local stereology ([10]) where the aim is to estimate quantitative properties of a spatial structure from random sections passing through reference points. A model example is the case where the structure is a biological cell and the reference point is the cell nucleus or some identifiable part of the nucleus such as a nucleolus. In this example, the cell is regarded as a neighbourhood of its nucleus. Some of the methods to be described in these lecture notes are widely used and in fact highly cited in the biomedical literature, see e.g. [7, 11, 20].

I also want to use the opportunity to sincerely thank Professor Andreas Bernig for giving me the opportunity to give these Erasmus lectures about rotational integral geometry at Goethe University, Frankfurt.

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Eva B. Vedel Jensen

Overview of lectures

Lecture 1: Rotational integrals of intrinsic volumes
We will study rotational integrals of sectional volume, sectional surface area and, more generally, intrinsic volumes. Rotational integrals of sectional volume are easily treated, using the Blaschke-Petkantschin formula. The case of sectional surface area and the remaining intrinsic volumes is more involved.

Lecture 2: Intrinsic volumes as rotational integrals
In this lecture, we study the 'opposite/inverse' problem of determining the measurement in the section with rotational integral equal to a given intrinsic volume. We derive a functional, defined on the section, with this property and study its geometric properties.

Lecture 3: Rotational integral geometry of tensors
We will extend the results of the previous lectures to Minkowski tensors.

Lecture 4: Future topics in rotational integral geometry
We will discuss a number of important open problems in rotational integral geometry, including uniqueness of the functional defined in Lecture 2, estimation of tensors in particle populations and a principal rotational formula.

Literature:
As background material, we recommend the book [10] on local stereology. Most of the material presented in this lecture series is also described in the papers [1, 2, 3, 12].

Lecture 1:
Rotational integrals of intrinsic volumes

We let $\mathcal{K}(\mathbb{R}^d)$ denote the set of convex bodies (compact and convex sets) in $\mathbb{R}^d$. The geometric identities considered throughout these lecture notes have the following general form

$$\int \alpha(X \cap L)\mathrm{d}L = \beta(X), \tag{1}$$

where α, β are geometrical quantities to be defined more precisely below, $X \in \mathcal{K}(\mathbb{R}^d)$ is the spatial object of interest, L is the probe (line, plane, grid of parallel lines, ...) and $\mathrm{d}L$ is 'uniform integration' over positions of L. In rotational integral geometry, we focus on geometric identities for j-dimensional planes L_j in $\mathbb{R}^d$ passing through the origin O (L_j is a j-dimensional linear subspace in $\mathbb{R}^d$, called a j-subspace in the following). The choice of origin is an important question in applications; in biomedicine, X is typically a cell and O is the nucleus or a nucleolus of the cell.

In this lecture, rotational integrals of intrinsic volumes will be studied. So α is an intrinsic volume, determined on the section, and the aim is to find the corresponding β. As we shall see, β involves weighted curvature measures. Intrinsic volumes are special cases of the Minkowski tensors to be considered in Lecture 3.

Recal that for $X \in \mathcal{K}(\mathbb{R}^d)$, we can define $d+1$ intrinsic volumes $V_k(X)$, $k = 0, \dots, d$. We have

- $V_d(X) =$ volume (Lebesgue measure) of X
- $V_{d-1}(X) = 2^{-1}\times$ surface area of X
- $V_0(X) =$ the Euler-Poincaré characteristic of X

For non-empty $X \in \mathcal{K}(\mathbb{R}^d)$, V_0 is thus identically equal to 1. The intrinsic volumes can be extended to larger set classes for which V_0 contains interesting topological information.

The intrinsic volumes are examples of real-valued valuations on $\mathbb{R}^d$. They are motion invariant and continuous with respect to the Hausdorff metric. Recall that a real-valued valuation on $\mathbb{R}^d$ is a mapping $\mu : \mathcal{K}(\mathbb{R}^d) \to \mathbb{R}$ satisfying

$$\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L),$$

whenever $K, L, K \cup L \in \mathcal{K}(\mathbb{R}^d)$. Hadwiger's famous characterization theorem states that any motion invariant, continuous valuation is a linear combination of intrinsic volumes.

The classical Crofton formula relates intrinsic volumes defined on j-dimensional affine subspaces to intrinsic volumes of the original set

$$\int_{\mathcal{F}_j^d} V_k(X \cap F_j)\mathrm{d}F_j^d = c_{d,j,k} V_{d-j+k}(X), \tag{2}$$
$$j = 0, 1, \dots, d, k = 0, 1, \dots, j.$$

Here, $\mathcal{F}_j^d$ is the set of j-dimensional affine subspaces in $\mathbb{R}^d$. Any $F_j \in \mathcal{F}_j^d$ is of the form $F_j = x + L_j$, where L_j is a j-subspace and $x \in L_j^\perp$. Furthermore,

$dF_j^d = \lambda_{d-j}(dx)\, dL_j^d$, where dL_j^d is the element of the rotation invariant measure on the set $\mathcal{L}_j^d$ of $j-$subspaces in $\mathbb{R}^d$ and λ_{d-j} is the Lebesgue measure on $L_j^\perp$. In these notes, the rotation invariant measure is normalized such that

$$\int dL_j^d = c_{d,j} = \omega_d \cdots \omega_{d-j+1} / \omega_j \cdots \omega_1. \tag{3}$$

Here, $\omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of unit ball in $\mathbb{R}^d$. In (2), $c_{d,j,k}$ is a known constant

$$c_{d,j,k} = c_{d,j} \cdot \frac{\Gamma(\frac{j+1}{2})\Gamma(\frac{d+k-j+1}{2})}{\Gamma(\frac{k+1}{2})\Gamma(\frac{d+1}{2})}. \tag{4}$$

Note that for $k = j$, the Crofton formula relates Lebesgue measure on sections $X \cap F_j$ to Lebesgue measure of the original set X. Likewise, for $k = j - 1$, sectional surface area is related to the surface area of X.

In rotational integral geometry, the interest is instead in rotational averages of intrinsic volumes

$$\int_{\mathcal{L}_j^d} V_k(X \cap L_j) dL_j^d = ?$$
$$j = 1, \ldots, d, k = 0, \ldots, j.$$

These integrals are valuations on $\mathbb{R}^d$. They are rotation invariant, but typically not translation invariant.

Let us first consider the case $k = j$. This is the simplest case where Lebesgue measure is measured on the section. For $j = 1, \ldots, d$, we have

$$\int_{\mathcal{L}_j^d} V_j(X \cap L_j) dL_j^d = c_{d-1,j-1} \int_X |x|^{-(d-j)} \lambda_d(dx). \tag{5}$$

The proof of this result is based on the Blaschke-Petkantschin formula. This formula exists in many versions. Generally, the Blaschke-Petkantschin formula concerns a decomposition of a product of Hausdorff measures, see [10, Theorem 5.6]. Here, we only need the decomposition of a single copy of Lebesgue measure. In this case, the Blaschke-Petkantschin formula takes the following form

$$c_{d-1,j-1}\lambda_d(dx) = |x|^{d-j} \lambda_j(dx) dL_j^d,$$

see [10, Proposition 4.5]. For $j = 1$ (line sections), the Blaschke-Petkantschin formula is simply polar decomposition in $\mathbb{R}^d$.

Example 1.1. For $d = 3$ and $j = 2$, we get, cf. (5),

$$\int_{\mathcal{L}_2^3} \text{area}(X \cap L_2)\, dL_2^3 = \beta(X),$$

where

$$\beta(X) = \pi \int_X |x|^{-1} \lambda_3(dx). \qquad \square$$

The situation is much more complicated, when $k < j$. Assume for simplicity of the presentation that X is a compact $d-$dimensional C^2 manifold with boundary. Then, under mild regularity conditions,

$$
\begin{aligned}
&\int_{\mathcal{L}_j^d} V_k(X \cap L_j)\,\mathrm{d}L_j^d \\
&\quad = \int_{\partial X} |x|^{-(d-j)} \sum_{|I|=j-1-k} w_{I,j,k}(x) \prod_{i\in I} \kappa_i(x) \mathcal{H}^{d-1}(\mathrm{d}x),
\end{aligned} \tag{6}
$$

where ∂X is the boundary of X, the sum runs over all subsets of $\{1,\dots,d-1\}$ with $j-1-k$ elements, the $w_{I,j,k}$s are weight functions involving hypergeometric functions, $\kappa_i(x)$, $i = 1,\dots,d-1$, are the principal curvatures at $x \in \partial X$ and $\mathcal{H}^{d-1}$ is Hausdorff measure. In 2008, this result was published in *Adv. Appl. Math.* by Jensen and Rataj ([12]). Here, the result was established for the more general set class consisting of sets with positive reach. The proof involves extensive geometric measure theory.

Very recently, the explicit form of the weight functions $w_{I,j,k}$ has been published ([2]). Note that if X is a ball centred at the origin O, then the $w_{I,j,k}$s are constant and $|x|$ is also constant when $x \in \partial X$. We are back to the classical Crofton formula. Generally, the $w_{I,j,k}$s depend on the angle between x and the outer unit normal $u(x)$ at $x \in \partial X$, *and* the angle between x and the subspace spanned by the principal directions with indices outside I. In [2], it is shown for $k < j$ that $\int_{\mathcal{L}_j^d} V_k(X\cap L_j)\mathrm{d}L_j^d$ can be expressed as an integral with respect to flag measures.

The special case $k = j-1$ gives rise to some simplifications of (6). When $k = j-1$, $I = \emptyset$, the sum on the right-hand side of (6) has only one element and the curvature product disappears. The following result holds for the rotational average of the sectional surface area

$$
\begin{aligned}
&\int_{\mathcal{L}_j^d} V_{j-1}(X \cap L_j)\,\mathrm{d}L_j^d \\
&\quad = \frac{c_{d-1,j-1}}{2} \int_{\partial X} |x|^{-(d-j)} F_{-\frac{1}{2},\frac{d-j}{2};\frac{d-1}{2}}(\sin^2\beta(x))\mathcal{H}^{d-1}(\mathrm{d}x),
\end{aligned}
$$

where F is a hypergeometric function and $\beta(x)$ is the angle between $x \in \partial X$ and the unique outer unit normal $u(x)$ to the boundary at $x \in \partial X$ (unique because of the smoothness condition).

The class of hypergeometric functions is parametrized by three parameters and has well-known series expansions as well as integral representations. In particular, we have for $0 < \beta < \gamma$ the following integral representation

$$
F_{\alpha,\beta;\gamma}(z) = \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 (1-zy)^{-\alpha} y^{\beta-1}(1-y)^{\gamma-\beta-1}\mathrm{d}y. \tag{7}
$$

Example 1.2. For $d = 3$ and $j = 2$, we find, using (7),

$$
\begin{aligned}
F_{-\frac{1}{2},\frac{d-j}{2};\frac{d-1}{2}}(\sin^2\beta(x)) &= \frac{2}{\pi}\int_0^{\pi/2} (1-\sin^2\beta(x)\sin^2\phi)^{1/2}\mathrm{d}\phi \\
&= \frac{2}{\pi} E(|\sin\beta(X)|, \pi/2),
\end{aligned}
$$

where E is the elliptic integral of the second kind. We find

$$
\int_{\mathcal{L}_2^3} \mathrm{length}(\partial X \cap L_2)\,\mathrm{d}L_2^3 = \beta(X),
$$

where

$$\beta(X) = 2 \int_{\partial X} |x|^{-1} E(|\sin\beta(X)|, \pi/2)\, \mathcal{H}^2(\mathrm{d}x). \qquad \square$$

Lecture 2: Intrinsic volumes as rotational integrals

In this lecture, we want to study the 'opposite/inverse' problem of determining the measurement in the section with rotational integral equal to a given intrinsic volume. So now β is an intrinsic volume and the aim is to find α such that (1) is satisfied. This problem has been studied in detail in [1, 6].

More specifically, we want to find a functional $\alpha_{j,k}$, satisfying the following rotational integral equation

$$\int_{\mathcal{L}_j^d} \alpha_{j,k}(X \cap L_j)\mathrm{d}L_j^d = V_{d-j+k}(X), \tag{8}$$

$j = 1, \ldots, d$, $k = 1, \ldots, j$. From an applied point of view, this question is more interesting than the one studied in the previous lecture, because $\alpha_{j,k}$ is then the measurement you need to perform in the section. This measurement has a rotational average equal to the intrinsic volume considered and can be used to estimate the intrinsic volume.

Let us first consider a simple example in $\mathbb{R}^2$ with $d = 2$ and $j = k = 1$. The aim is then to find a functional $\alpha_{1,1}$ such that

$$\int_{\mathcal{L}_1^2} \alpha_{1,1}(X \cap L_1)\mathrm{d}L_1^2 = \text{area}\,(X). \tag{9}$$

It is fairly easy to find a solution to this problem. Consider an infinitesimal neighbourhood of $x \in X$ of area $\lambda_2(\mathrm{d}x)$. Transforming to polar coordinates in $\mathbb{R}^2$, $x = (r\cos\theta, r\sin\theta)$, gives us the following decomposition of area measure in the plane

$$\lambda_2(\mathrm{d}x) = |r|\mathrm{d}r\mathrm{d}\theta, \tag{10}$$

$r \in \mathbb{R}$, $\theta \in [0, \pi)$. Identifying θ with the line L_1 passing through the origin, having an angle θ with a fixed axis, we have $\mathrm{d}L_1^2 = \mathrm{d}\theta$, and (10) can equivalently be expressed as

$$\lambda_2(\mathrm{d}x) = d(x, O)\, \lambda_1(\mathrm{d}x)\mathrm{d}L_1^2,$$

where λ_1 is Lebesgue measure on L_1 and $d(\cdot, \cdot)$ is the notation used for Euclidean distance. It follows that

$$\alpha_{1,1}(X \cap L_1) = \int_{X \cap L_1} d(x, O)\, \lambda_1(\mathrm{d}x)$$

is a solution to (9).

A solution to the general problem of finding a functional $\alpha_{j,k}$ satisfying (8) can be derived by combining the classical Crofton formula with a geometric measure decomposition of the motion invariant measure of r-dimensional affine subspace in $\mathbb{R}^d$, see [19, p. 285],

$$\mathrm{d}F_r^d = d(F_r, O)^{d-r-1}\mathrm{d}F_r^{r+1}\mathrm{d}L_{r+1}^d, \tag{11}$$

$r = 0, \ldots, d-1$. The solution is given in the proposition below. We will generalize this result to Minkowski tensors in Lecture 3.

Proposition 2.1 (Auneau and Jensen, 2010 [1]; Gual-Arnau and Cruz-Orive, 2010 [6]). *Let Y be a compact and convex subset of $\mathbb{R}^j$. For $j = 1, \ldots, d$, $k = 1, \ldots, j$, the functional*

$$\alpha_{j,k}(Y) = \frac{1}{c_{d,j-1,k-1}} \int_{\mathcal{F}^j_{j-1}} d(O, F_{j-1})^{d-j} V_{k-1}(Y \cap F_{j-1}) \mathrm{d}F^j_{j-1}$$

is a solution to (8).

Proof. Using the measure decompostion (11), we find

$$\begin{aligned}
&\int_{\mathcal{L}^d_j} \alpha_{j,k}(X \cap L_j) \mathrm{d}L^d_j \\
&= \frac{1}{c_{d,j-1,k-1}} \int_{\mathcal{L}^d_j} \int_{\mathcal{F}^j_{j-1}} d(O, F_{j-1})^{d-j} V_{k-1}(X \cap L_j \cap F_{j-1}) \mathrm{d}F^j_{j-1} \mathrm{d}L^d_j \\
&= \frac{1}{c_{d,j-1,k-1}} \int_{\mathcal{L}^d_j} \int_{\mathcal{F}^j_{j-1}} d(O, F_{j-1})^{d-(j-1)-1} V_{k-1}(X \cap F_{j-1}) \mathrm{d}F^j_{j-1} \mathrm{d}L^d_j \\
&= \frac{1}{c_{d,j-1,k-1}} \int_{\mathcal{F}^d_{j-1}} V_{k-1}(X \cap F_{j-1}) \mathrm{d}F^d_{j-1} \\
&= V_{d-j+k}(X).
\end{aligned}$$

At the last equality sign, we have used the Crofton formula (2). □

Example 2.2. For $d = 3$ and $j = k = 2$, we get

$$\int_{\mathcal{L}^3_2} \alpha_{2,2}(X \cap L_2) \mathrm{d}L^3_2 = \lambda_3(X),$$

where

$$\alpha_{2,2}(X \cap L_2) = \frac{1}{c_{3,1,1}} \int_{\mathcal{F}^2_1} d(O, F_1) \, \mathrm{length}(X \cap F_1) \mathrm{d}F^2_1.$$

Furthermore, for $d = 3$, $j = 2$ and $k = 1$, we get

$$\int_{\mathcal{L}^3_2} \alpha_{2,1}(X \cap L_2) \mathrm{d}L^3_2 = \frac{1}{2} \text{surface area}(X),$$

where

$$\alpha_{2,1}(X \cap L_2) = \frac{1}{c_{3,1,0}} \int_{\mathcal{F}^2_1} d(O, F_1) \, \mathbf{1}\{X \cap F_1 \neq \emptyset\} \mathrm{d}F^2_1.$$

□

It was shown in [1] that for $k = j$ and $k = j-1$ the functional $\alpha_{j,k}$ can be considerably simplified and given in more explicit form. The result is presented in the corollary below.

Corollary 2.3. *Let the situation be as in Proposition 2.1. Then,*

$$\alpha_{j,j}(Y) = \frac{1}{c_{d-1,j-1}} \int_Y |z|^{d-j} \lambda_j(\mathrm{d}z)$$

and

$$2c_{d-1,j-1}\,\alpha_{j,j-1}(Y) = \int_{\partial Y} |z|^{d-j} F_{-\frac{1}{2},-\frac{d-j}{2};\frac{j-1}{2}}(\sin^2(\beta(z)))\mathcal{H}^{j-1}(\mathrm{d}z),$$

where $\beta(z)$ is the angle between $z \in \partial Y$ and the unique outer unit normal $u(z)$ to the boundary of Y at $z \in \partial Y$.

Proof. Using that $F_{j-1} = L_{j-1} + x$, where $x \in L_{j-1}^{\perp}$, we find

$$\begin{aligned}
\alpha_{j,j}(Y) &= \frac{1}{c_{d,j-1,j-1}} \int_{\mathcal{F}_{j-1}^j} d(O, F_{j-1})^{d-j} V_{j-1}(Y \cap F_{j-1})\mathrm{d}F_{j-1}^j \\
&= \frac{1}{c_{d,j-1,j-1}} \int_{\mathcal{L}_{j-1}^j} \int_{L_{j-1}^{\perp}} |x|^{d-j} V_{j-1}(Y \cap (L_{j-1}+x))\lambda_1(\mathrm{d}x)\mathrm{d}L_{j-1}^j \\
&= \frac{1}{c_{d,j-1,j-1}} \int_{\mathcal{L}_{j-1}^j} \int_{L_{j-1}^{\perp}} \int_{Y \cap (L_{j-1}+x)} |x|^{d-j} \lambda_{j-1}(\mathrm{d}y)\lambda_1(\mathrm{d}x)\mathrm{d}L_{j-1}^j \\
&= \frac{1}{c_{d,j-1,j-1}} \int_{\mathcal{L}_{j-1}^j} \int_Y |p(z|L_{j-1}^{\perp})|^{d-j} \lambda_j(\mathrm{d}z)\mathrm{d}L_{j-1}^j \\
&= \frac{1}{c_{d,j-1,j-1}} \int_Y |z|^{d-j} \left(\int_{\mathcal{L}_{j-1}^j} \frac{|p(z|L_{j-1}^{\perp})|^{d-j}}{|z|^{d-j}} \mathrm{d}L_{j-1}^j \right) \lambda_j(\mathrm{d}z) \\
&= \frac{1}{c_{d,j-1,j-1}} \int_Y |z|^{d-j} \left(\frac{c_{j,j-1}}{B(\frac{1}{2},\frac{j-1}{2})} \int_0^1 y^{\frac{d-j-1}{2}}(1-y)^{\frac{j-3}{2}}\mathrm{d}y \right) \lambda_j(\mathrm{d}z).
\end{aligned}$$

At the last equality sign, we have used [10, Proposition 3.9]. The result concerning $\alpha_{j,j}$ now follows immediately, using (3) and (4).

The result concerning $\alpha_{j,j-1}$ is more difficult to show. The details can be found in [1]. Let us here just give a proof sketch. In [1], it is shown that

$$c_{d,j-1,j-2}\alpha_{j,j-1}(Y) = \frac{1}{2} \int_{\partial Y} \int_{\mathcal{L}_{j-1}^j} |p(u(z)|L_{j-1})|\,|p(z|L_{j-1}^{\perp})|^{d-j}\mathrm{d}L_{j-1}^j \mathcal{H}^{j-1}(\mathrm{d}z).$$

The result now follows if we use the following result proved in [1]. For x and y unit vectors in L_j and non-negative integers n, m, we have

$$\begin{aligned}
&\int_{\mathcal{L}_{j-1}^j} |p(x|L_{j-1})|^m\,|p(y|L_{j-1}^{\perp})|^n \mathrm{d}L_{j-1}^j \\
&= \frac{\omega_{j-1}}{2} B\left(\frac{n+1}{2}, \frac{m+j-1}{2}\right) F_{-\frac{m}{2},-\frac{n}{2};\frac{j-1}{2}}(\sin^2 \angle(x,y)).
\end{aligned}$$

□

In this lecture we have found a functional $\alpha_{j,k}$ satisfying the rotational integral equation (8). A natural question to ask is whether $\alpha_{j,k}$ is unique. If a solution is seeked among rotation invariant functionals only, this is indeed the case for $j = k = 1$, cf. [13]. It is an open question whether uniqueness holds for general j and k.

Lecture 3:
Rotational integral geometry of tensors

In this lecture, we will extend the results of the previous two lectures to tensor valuations. These results are very recent ([3]). We will define so-called integrated Minkowski tensors for which a genuine rotational Crofton formula holds. As we shall see, using integrated Minkowski tensors, the two problems of finding (1) rotational averages of intrinsic volumes and (2) expressing intrinsic volumes as rotational integrals can be given a common formulation.

For non-negative integers r and s, $k = 0, \ldots, d-1$, the Minkowski tensors are

$$\Phi_{k,r,s}(X) := \frac{\omega_{d-k}}{r!\, s!\, \omega_{d-k+s}} \int_{\mathbb{R}^d \times S^{d-1}} x^r u^s \Lambda_k(X, \mathrm{d}(x,u)) \text{ (surface tensor)}$$

$$\Phi_{d,r,0}(X) := \frac{1}{r!} \int_X x^r \lambda_d(\mathrm{d}x) \text{ (volume tensor)}$$

Here, x^r is the symmetric tensor of rank r determined by x, while $x^r u^s$ is the symmetric tensor product of x^r and u^s. Furthermore, $\Lambda_k(X, \cdot)$ is the kth support measure or generalized curvature measure of X, $k = 0, \ldots, d-1$. The support measure Λ_k is concentrated on the normal bundle $\mathrm{Nor}X$ of X which consists of all pairs (x, u) where $x \in \partial X$ and u is an outer normal vector of X at x. For $r = s = 0$, we have $\Phi_{k,0,0} = V_k$, the kth intrinsic volume, $k = 0, \ldots, d$.

For the development of rotational integral geometry of Minkowski tensors, we will now introduce the integrated Minkowski tensors. These tensors are weighted integrals of Minkowski tensors defined on j-dimensional affine subspaces.

Definition 3.1. *For $0 \le k < j < d$, $t > j - d$ and non-negative integers r and s, the integrated Minkowski tensors are*

$$\Phi^{j,t}_{k,r,s}(X) := \int_{\mathcal{F}^d_j} \Phi^{(F_j)}_{k,r,s}(X \cap F_j) d(F_j, O)^t \mathrm{d}F^d_j,$$

and

$$\Phi^{j,t}_{j,r,0}(X) := \int_{\mathcal{F}^d_j} \Phi^{(F_j)}_{j,r,0}(X \cap F_j) d(F_j, O)^t \mathrm{d}F^d_j,$$

where the integrands $\Phi^{(F_j)}_{k,r,s}(X \cap F_j)$ and $\Phi^{(F_j)}_{j,r,0}(X \cap F_j)$ are calculated relative to F_j. The condition $t > j - d$ ensures that $\Phi^{j,t}_{k,r,s}(X)$ is well-defined.

There are a number of interesting special cases of integrated Minkowski tensors. Using Definition 3.1 for $j = d$ and $t = 0$ gives $\Phi^{d,0}_{k,r,s} = \Phi_{k,r,s}$. Furthermore,

$$\Phi^{j,0}_{k,0,0} \propto V_{d+k-j}, \quad 0 \le k \le j < d \quad \text{(classical Crofton formula)}$$

More generally, using [9, Theorem 2.4 and 2.5], we find

$$\Phi^{j,0}_{k,r,s} \propto \Phi_{d+k-j,r,s}, \quad 0 \le k < j < d, s = 0, 1, \tag{12}$$

$$\Phi^{j,0}_{j,r,0} \propto \Phi_{d,r,0}, \qquad 0 < j < d, \tag{13}$$

where $\propto$ in (12) and (13) means that the two functionals are identical up to a known constant. In [9], it is also shown for arbitrary non-negative integers s that $\Phi^{j,0}_{k,r,s}$ is a linear combination of Minkowski tensors.

The integrated Minkowski tensors obey a genuine rotational Crofton formula.

Proposition 3.2. *For $0 \leq k < j < p \leq d$, $t > j - d$ and non-negative integers r and s, we have*

$$\Phi_{k,r,s}^{j,t}(X) = \frac{1}{c_{d-j-1,p-j-1}} \int_{\mathcal{L}_p^d} \Phi_{k,r,s}^{j,d-p+t}(X \cap L_p) \mathrm{d}L_p^d. \tag{14}$$

For $j = k$, (14) holds for $s = 0$.

Proof. We use the following decomposition

$$\mathrm{d}F_j^d = \frac{d(F_j, O)^{d-p}}{c_{d-j-1,p-j-1}} \, \mathrm{d}F_j^p \, \mathrm{d}L_p^d,$$

$0 < j < p \leq d$, see [19, p. 285]. We find

$$\begin{aligned}
&\Phi_{k,r,s}^{j,t}(X) \\
&= \int_{\mathcal{F}_j^d} \Phi_{k,r,s}^{(F_j)}(X \cap F_j) d(F_j, O)^t \mathrm{d}F_j^d \\
&= \frac{1}{c_{d-j-1,p-j-1}} \int_{\mathcal{L}_p^d} \int_{\mathcal{F}_j^p} \Phi_{k,r,s}^{(F_j)}(X \cap F_j) d(F_j, O)^{d-p+t} \mathrm{d}F_j^p \, \mathrm{d}L_p^d \\
&= \frac{1}{c_{d-j-1,p-j-1}} \int_{\mathcal{L}_p^d} \Phi_{k,r,s}^{j,d-p+t}(X \cap L_p) \mathrm{d}L_p^d.
\end{aligned}$$

The second statement is proved in exactly the same manner. □

By choosing the parameters in the rotational Crofton formula appropriately, either the left-hand side or the right-hand side of the formula becomes a classical Minkowski tensor.

Corollary 3.3 (rotational averages of Minkowski tensors). *For $s \in \{0, 1\}$ and $t = p - d$, the result in Proposition 3.2 reduces to*

$$\int_{\mathcal{L}_p^d} \Phi_{m,r,s}^{(L_p)}(X \cap L_p) \, \mathrm{d}L_p^d \propto \Phi_{m-q,r,s}^{p-q,p-d}(X), \tag{15}$$

for $0 < q \leq m < p \leq d$.

If $m = p$, then $s = 0$, and we get

$$\int_{\mathcal{L}_p^d} \Phi_{p,r,0}^{(L_p)}(X \cap L_p) \, \mathrm{d}L_p^d \propto \Phi_{p-q,r,0}^{p-q,p-d}(X), \tag{16}$$

for $0 < q < p \leq d$.

Proof. Combining Proposition 3.2 with equation (12), we find

$$\begin{aligned}
\int_{\mathcal{L}_p^d} \Phi_{m,r,s}^{(L_p)}(X \cap L_p) \, \mathrm{d}L_p^d \quad &\propto \quad \int_{\mathcal{L}_p^d} \Phi_{m-q,r,s}^{p-q,0}(X \cap L_p) \, \mathrm{d}L_p^d \\
&\propto \quad \Phi_{m-q,r,s}^{p-q,p-d}(X).
\end{aligned}$$

The second statement is proved in exactly the same manner. □

Note that for $r = s = 0$, the left-handsides of (15) and (16) take the form of a rotational average of an intrinsic volume, see Lecture 1 and [2, 12].

From an applied point of view, it is in fact more interesting to try to find the functional defined on the subspace L_p whose rotational average equals a given classical Minkowski tensor. This problem can again be solved for $s \in \{0,1\}$ by combining Proposition 3.2 with equations (12) and (13).

Corollary 3.4 (Minkowski tensors as rotational averages). *For $s \in \{0,1\}$ and $t = 0$, the result in Proposition 3.2 reduces to*

$$\Phi_{d+m-p,r,s}(X) \propto \int_{\mathcal{L}_p^d} \Phi_{m-q,r,s}^{p-q,d-p}(X \cap L_p)\mathrm{d}L_p^d, \tag{17}$$

for $0 < q \le m < p \le d$.

If $m = p$, then $s = 0$, and we get

$$\Phi_{d,r,0}(X) \propto \int_{\mathcal{L}_p^d} \Phi_{p-q,r,0}^{p-q,d-p}(X \cap L_p)\mathrm{d}L_p^d, \tag{18}$$

for $0 < q < p \le d$.

Proof. Combining Proposition 3.2 with equation (12), we find

$$\begin{aligned} \int_{\mathcal{L}_p^d} \Phi_{m-q,r,s}^{p-q,d-p}(X \cap L_p)\mathrm{d}L_p^d \quad &\propto \quad \Phi_{m-q,r,s}^{p-q,0}(X) \\ &\propto \quad \Phi_{d+m-p,r,s}(X). \end{aligned}$$

The second statement is proved in exactly the same manner. $\square$

For $r = s = 0$ and $q = 1$, the result in Corollary 3.4 reduces to the main result in [1].

It is clearly of interest to study what kind of geometric information the integrated Minkowski tensors carry about the original set X. In the proposition below, we give such geometric interpretation for $\Phi_{j,r,0}^{j,t}$ and $\Phi_{d-2,r,0}^{d-1,t}$. For a proof, the reader is referred to [3].

Proposition 3.5. *For $0 < j < d$, $t > j - d$ and a non-negative integer r*

$$\Phi_{j,r,0}^{j,t}(X) = \frac{c_{d,j}}{r!} \frac{\Gamma(\frac{t+d-j}{2})\Gamma(\frac{d}{2})}{\Gamma(\frac{t+d}{2})\Gamma(\frac{d-j}{2})} \int_X x^r |x|^t \lambda_d(\mathrm{d}x).$$

Furthermore, if X is a compact d-dimensional C^2 manifold with boundary, then for $t > 0$ and a non-negative integer r

$$\begin{aligned} \Phi_{d-2,r,0}^{d-1,t}(X) \quad &= \quad \frac{\omega_{d-1}}{4r!} B\left(\tfrac{t+1}{2}, \tfrac{d}{2}\right) \\ &\quad \times \int_{\partial X} x^r |x|^t F_{-\frac{1}{2},-\frac{t}{2};\frac{d-1}{2}}(\sin^2 \beta(x))\, \mathcal{H}^{d-1}(\mathrm{d}x). \end{aligned}$$

In Lecture 2, we studied the functional $\alpha_{j,k}$, see Proposition 2.1. Note that this functional is a special case of an integrated Minkowski tensor since $\alpha_{j,k} \propto \Phi_{k-1,0,0}^{j-1,d-j}$. Note also that, in Proposition 2.1, $\alpha_{j,k}$ is expressed as a functional defined on sets in $\mathbb{R}^j$. If we in Proposition 3.5 insert these parameter values, we get the result in Corollary 2.3.

Lecture 4:
Future topics in rotational integral geometry

The geometric identities we have considered in the previous lectures are all of the form

$$\int_{\mathcal{L}_p^d} \alpha(X \cap L_p) \mathrm{d}L_p^d = \beta(X), \tag{19}$$

where either α or β is an intrinsic volume or, more generally, a Minkowski tensor. In this lecture, we will discuss a number of topics for future research in rotational integral geometry, including uniqueness of α functionals, estimation of tensors in particle populations and a principal rotational formula.

Uniqueness

In Lecture 2, we have found a functional $\alpha_{p,k}$, satisfying

$$\int_{\mathcal{L}_p^d} \alpha_{p,k}(X \cap L_p) \mathrm{d}L_p^d = V_{d-p+k}(X), \quad X \in \mathcal{K}(\mathbb{R}^d), \tag{20}$$

$p = 1, \ldots, d$, $k = 1, \ldots, p$. The functional $\alpha_{p,k}$ is rotation invariant.

A natural question to ask is whether $\alpha_{p,k}$ is unique in the class of rotation invariant functionals, satisfying (20). The functional $\alpha_{p,k}$ is defined on the set

$$\mathcal{S}_p = \{X \cap L_p : X \in \mathcal{K}(\mathbb{R}^d), L_p \in \mathcal{L}_p^d\}$$

In the following, we will restrict attention to convex bodies containing the origin, i.e. $O \in X$.

In the case $p = k = 1$, $\alpha_{p,k}$ reduces to

$$\begin{aligned} \alpha_{1,1}(X \cap L_1) &= \frac{1}{c_{d,0,0}} \int_{\mathcal{F}_0^1} d(O, F_0)^{d-1} V_0(X \cap L_1 \cap F_0) \mathrm{d}F_0^1 \\ &= \int_{X \cap L_1} d(O, x)^{d-1} \lambda_1(\mathrm{d}x). \end{aligned}$$

Note that since X is convex and $O \in X$, $X \cap L_1$ is of the form $[r(-u), Ru]$ where $r, R \geq 0$ and $u \in \mathbb{S}^{d-1}$ is chosen such that $L_1 = \mathrm{span}\{u\}$. In [13], it is shown that $\alpha_{1,1}$ is indeed the unique rotation invariant functional on $\mathcal{S}_1$, satisfying

$$\int_{\mathcal{L}_1^d} \alpha(X \cap L_1) \mathrm{d}L_1^d = V_d(X).$$

The situation for general p and k is still open. Very recently, some progress has been made in [13] for functionals of a particular form.

Stereology of tensors

The new geometric identities for Minkowski tensors can be used to estimate the distribution of a Minkowski tensor in a particle population from sectional data, thereby providing information about the orientation and shape of the particles. Let us assume that the particles are a realization of a marked point process

$\Psi = \{[x_i; \Xi_i]\}$ where the x_is are the points in $\mathbb{R}^d$ and the marks Ξ_i are convex and compact subsets of $\mathbb{R}^d$. The ith particle of the process is represented by $X_i = x_i + \Xi_i$. If the particle process is stationary, it can be shown for any non-negative measurable function h that

$$E \sum_i h(x_i, \Xi_i) = \lambda \int_{\mathbb{R}^d} \int_{\mathcal{K}^d} h(x, K) P_m(\mathrm{d}K) \lambda_d(\mathrm{d}x),$$

where λ is the particle intensity and P_m is a probability distribution on $\mathcal{K}(\mathbb{R}^d)$, called the particle distribution. We let Ξ_0 be a random convex and compact subset of $\mathbb{R}^d$ with distribution P_m.

Our aim is to estimate the distribution of $\beta(\Xi_0)$ from sectional data where β is a Minkowski tensor. Available for observation is a sample of particles $\{x_i + \Xi_i : x_i \in W\}$ collected in a sampling window. It is possible to perform measurements on any virtual section $\Xi_i \cap L_p$. If L_p is an isotropic section, then

$$E(\alpha(\Xi_i \cap L_p)|\Xi_i) = \int_{\mathcal{L}_p} \alpha(\Xi_i \cap L_p) \frac{\mathrm{d}L_p^d}{c_{d,p}} = \frac{1}{c_{d,p}} \beta(\Xi_i).$$

The distribution of $\beta(\Xi_0)$ can now be estimated by the empirical distribution of $\{\hat{\beta}(\Xi_i) : x_i \in W\}$, where

$$\hat{\beta}(\Xi_i) = c_{d,p} \sum_{j=1}^{N} \alpha(\Xi_i \cap L_{p,j})/N$$

and $L_{p,j}$, $j = 1, \ldots, N$, are replicated virtual isotropic sections. It still remains to study the statistical properties of the tensor estimators.

A principal rotational formula

To the best of our knowledge, a principal rotational formula is still not available in the literature. Focusing on intrinsic volumes, such a formula involves integrals of the form

$$\int_{SO_d} V_k(X \cap \rho Y) \nu(\mathrm{d}\rho), \tag{21}$$

$k = 0, \ldots, d$, where SO_d is the special orthogonal group in $\mathbb{R}^d$, X and Y are convex and compact subsets of $\mathbb{R}^d$, and ν is the unique rotation invariant probability measure on SO_d. From an applied point of view such a formula is interesting. Here, X is the unknown spatial structure of interest while Y is a known 'test set' constructed by the observer. The aim is to get information about X from observation of the intersection of X with a randomly rotated version of Y. For $k = d$, (21) is equal to

$$\frac{1}{\omega_d} \int_0^\infty r^{-(d-1)} \mathcal{H}^{d-1}(X \cap r\mathbb{S}^{d-1}) \mathcal{H}^{d-1}(Y \cap r\mathbb{S}^{d-1}) \, \mathrm{d}r.$$

To see this, we use that

$$\begin{aligned} \int_{SO_d} V_d(X \cap \rho Y) \nu(\mathrm{d}\rho) &= \int_{SO_d} \int_{\mathbb{R}^d} \mathbf{1}\{x \in X \cap \rho Y\} \lambda_d(\mathrm{d}x) \nu(\mathrm{d}\rho) \\ &= \int_{\mathbb{R}^d} \mathbf{1}\{x \in X\} \left[\int_{SO_d} \mathbf{1}\{x \in \rho Y\} \nu(\mathrm{d}\rho) \right] \lambda_d(\mathrm{d}x). \end{aligned}$$

Since

$$\begin{aligned}
\int_{SO_d} \mathbf{1}\{x \in \rho Y\}\nu(\mathrm{d}\rho) &= \int_{SO_d} \mathbf{1}\{\rho^{-1}x \in Y\}\nu(\mathrm{d}\rho) \\
&= \int_{SO_d} \mathbf{1}\{\rho x \in Y\}\nu(\mathrm{d}\rho) \\
&= \mathcal{H}^{d-1}(Y \cap |x|\,\mathbb{S}^{d-1})/\mathcal{H}^{d-1}(|x|\,\mathbb{S}^{d-1}) \\
&= |x|^{-(d-1)}\omega_d^{-1}\mathcal{H}^{d-1}(Y \cap |x|\,\mathbb{S}^{d-1}),
\end{aligned}$$

we obtain

$$\begin{aligned}
&\int_{SO_d} V_d(X \cap \rho Y)\nu(\mathrm{d}\rho) \\
&= \frac{1}{\omega_d}\int_X |x|^{-(d-1)}\mathcal{H}^{d-1}(Y \cap |x|\,\mathbb{S}^{d-1})\lambda_d(\mathrm{d}x) \\
&= \frac{1}{\omega_d}\int_0^\infty r^{-(d-1)}\mathcal{H}^{d-1}(X \cap r\mathbb{S}^{d-1})\mathcal{H}^{d-1}(Y \cap r\mathbb{S}^{d-1})\,\mathrm{d}r.
\end{aligned}$$

A result of a similar form involving two terms can be obtained for $k = d-1$. The case of general k is still open.

References

[1] Auneau, J. and Jensen, E.B.V. (2010): Expressing intrinsic volumes as rotational integrals. *Adv. Appl. Math.* **45**, 1-11.

[2] Auneau-Cognacq, J., Rataj, J. and Jensen, E.B.V. (2012): Closed form of the rotational Crofton formula. *Math. Nachr.* **285**, 164-180.

[3] Auneau-Cognacq, J., Ziegel, J. and Jensen, E.B.V. (2012): Rotational integral geometry of tensor valuations. *CSGB Research Report* **12-02**, Department of Mathematics, Aarhus University. Submitted.

[4] Beisbart, C., Barbosa, M.S., Wagner, H. and Costa, L. da F. (2006): Extended morphometric analysis of neuronal cells with Minkowski valuations. *Eur. Phys. J.* **B 52**, 531-546.

[5] Beisbart, C., Dahlke, R., Mecke, K.R. and Wagner, H. (2002): Vector- and tensor-valued descriptors for spatial patterns. In *Lecture Notes in Physics* **600**, pp. 249-271. Springer.

[6] Gual-Arnau, X., Cruz-Orive, L.M. and Nuno-Ballesteros, J.J. (2010): A new rotational integral formula for intrinsic volumes in space forms. *Adv. Appl. Math.* **44**, 298-308.

[7] Gundersen, H.J.G. (1988): The nucleator. *J. Microsc.* **151**, 3-21.

[8] Hansen-Goos, H. and Mecke, K. (2010): Tensorial density functional theory for non-spherical hard-body fluids. *J. Phys.: Condens. Matter* **22**, 1-16.

[9] Hug, D.,Schneider, R. and Schuster, R. (2008): Integral geometry of tensor valuations. *Adv. Appl. Math.* **41**, 482-509.

[10] Jensen, E.B.V. (1998): *Local Stereology.* World Scientific, Singapore.

[11] Jensen, E.B.V. and Gundersen, H.J.G. (1993): The rotator. *J. Microsc.* **170**, 35-44.

[12] Jensen, E.B.V. and Rataj, J. (2008): A rotational integral formula for intrinsic volumes. *Adv. Appl. Math.* **41**, 530-560.

[13] Kiderlen, M. *et al.* (2012): On uniqueness of rotational formulae. In preparation.

[14] Prokop, R.J. and Reeves, A.P. (1992): A survey of moment-based techniques for unoccluded object representation and recognition. *Graphical Models and Image Processing* **5**, 438-460.

[15] Schröder-Turk, G.E., Kapfer, S., Breidenbach, B., Beisbart, C. and Mecke, K. (2011): Tensorial Minkowski functionals and anisotropy measures for planar patterns. *J. Microsc.* **238**, 57-74.

[16] Schröder-Turk, G.E., Mickel, W., Kapfer, S.C., Klatt, M.A., Schaller, F.M., Hoffmann, M.J.F., Kleppmann, N., Armstrong, P., Inayat, A., Hug, D., Reichelsdorfer, M., Peukert, W., Schwieger, W. and Mecke, K. (2011): Minkowski tensor shape analysis of cellular, granular and porous structures. *Adv. Mater.* **23**, 2535-2553.

[17] Schneider, R. and Schuster, R. (2002): Tensor valuations on convex bodies and integral geometry, II. *Rend. Circ. Mat. Palermo (2) Suppl.* **70**, 295-314.

[18] Schneider, R. and Schuster, R. (2006): Particle orientation from section stereology. *Rend. Circ. Mat. Palermo (2) Suppl.* **77**, 623-633.

[19] Schneider, R. and Weil, W. (2008): *Stochastic and Integral Geometry.* Springer, Berlin.

[20] Tandrup, T., Gundersen, H.J.G. and Jensen, E.B.V. (1997): The optical rotator. *J. Microsc.* **186**, 108-120.