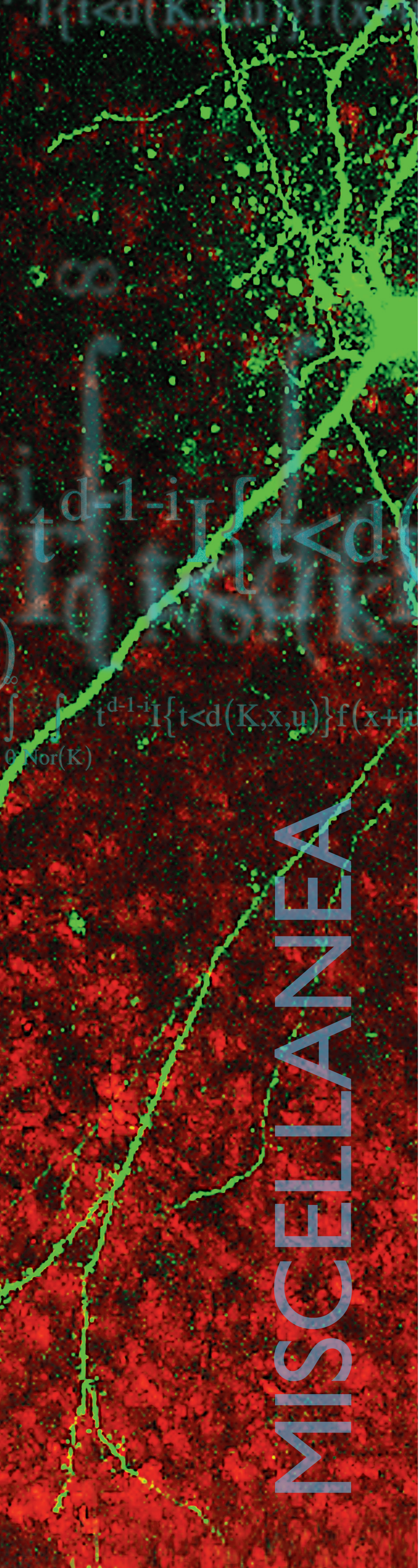

CENTRE FOR **STOCHASTIC GEOMETRY**
AND ADVANCED **BIOIMAGING**

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Course on Advanced Valuation Theory

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COURSE ON ADVANCED VALUATION THEORY

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Contents

Overview

These notes grew out of the Erasmus *Course on advanced valuation theory* given at the University of Aarhus from March 14 to March 17, 2011. I wish to thank all participants for many questions during the lectures and in particular Professor Eva Vedel Jensen for the invitation and the organization of this event.

Lecture 1: Translation invariant valuations
We will study the space of translation invariant continuous valuations. The first main theorem is McMullen's decomposition theorem which we will prove.

Lecture 2: The Klain embedding
We will prove Klain's characterization of the volume. As a corollary, we obtain a simple proof of Hadwiger's characterization theorem. Klain's theorem is also quite useful to describe even translation invariant valuations by functions on some Grassmannian manifold.

Lecture 3: Algebraic structures on the space of valuations
We formulate Alesker's irreducibility theorem and deduce some important corollaries like McMullen's conjecture. Then we study some geometrically

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natural algebraic structures on the space of translation invariant valuations: a graded product, a convolution product and a Fourier transform relating these two.

Lecture 4: Integral geometry of $SO(n)$ and other groups

If a group G acts transitively on the unit sphere bundle of a vector space V, then the space of translation invariant, G-invariant continuous valuations is finite-dimensional. The best known and classical case is $G = SO(n)$.

We will show in detail how the material from the preceding lectures shed new light on several well-known theorems like Hadwiger's characterization theorem and the principal kinematic formula. The main theorem is the *fundamental theorem of algebraic integral geometry*.

Literature: As background for lectures 1 and 4, we recommend the books by Schneider [15] and Klain and Rota [13]. Most of the material presented in this lecture series is also described in the two survey papers [10, 5].

1. Lecture 1: Translation invariant valuations

Let V be a finite-dimensional vector space and denote by $\mathcal{K}(V)$ the set of compact convex subsets of V. A valuation on V is a map $\mu : \mathcal{K}(V) \to \mathbb{C}$ which is finitely additive in the following sense:

$$\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L)$$

whenever $K, L, K \cup L \in \mathcal{K}(V)$.

A valuation μ is *continuous* if it is continuous with respect to the Hausdorff topology on $\mathcal{K}(V)$. It is *translation invariant* if $\mu(v + K) = \mu(K)$ for all $v \in V$.

Now suppose V is a Euclidean vector space of dimension n. Then μ is called *motion invariant* if $\mu(\bar{g}K) = \mu(K)$ for all Euclidean motions $\bar{g}$.

An example of a motion invariant valuation is the *intrinsic volume* μ_k ($k = 0, 1, \ldots, n$). It is defined by

$$\mu_k(K) := c_{n,k} \int_{\mathrm{Gr}_k V} \mathrm{vol}_k(\pi_L K) dL,$$

where $\pi_L : V \to L$ is the orthogonal projection and where the normalizing constant $c_{n,k}$ is chosen such that $\mu_k(K) = \mathrm{vol}_k(K)$ whenever K is k-dimensional. The measure dL on the Grassmannmanifold $\mathrm{Gr}_k V$ (which consists of all k-dimensional subspaces of V) is the Haar probability measure.

Hadwiger's famous characterization theorem (see Lecture 2) states that the space $\mathrm{Val}^{SO(n)}$ of motion invariant continuous valuations is of dimension $n + 1$, where $n = \dim V$. The only natural choice (up to scale) of a basis of $\mathrm{Val}^{SO(n)}$ consists of the intrinsic volumes $\mu_0, \ldots, \mu_n$. From Hadwiger's theorem, the array of kinematic formulas, mean projection formulas, additive kinematic formulas and many other results can be obtained in an elegant and simple way.

Looking at the definition of μ_k, it is clear that μ_k is *homogeneous of degree* k, i.e. $\mu_k(tK) = t^k\mu_k(K)$ for all $t > 0$. Intuitively, μ_k scales like a k-dimensional volume.

In general, a translation invariant valuation μ is called k*-homogeneous* or *of degree* k if $\mu(tK) = t^k\mu(K), t > 0$. As an example, let $\mu(K)$ be the k-dimensional volume of the orthogonal projection of K onto some fixed k-dimensional subspace $L \subset V$. Then μ is of degree k.

Let us introduce some notation. The space of translation invariant, continuous valuations on V is denoted by Val. The structure of Val and its relations to kinematic formulas is the main focus of these lectures. A first and trivial observation is that Val is a vector space. It is of infinite dimension (provided $n > 1$).

The subspace of k-homogeneous valuations in Val is denoted by Val_k. Hence $\mu_k \in \mathrm{Val}_k$.

The main result of the first lecture is the following theorem.

Theorem 1.1 (P. McMullen, 1977 [14])**.** *Each valuation* $\phi \in \mathrm{Val}$ *can be written uniquely as*

$$\phi = \phi_0 + \ldots + \phi_n$$

with $\deg \phi_k = k$. *In other words,*

$$\mathrm{Val} = \bigoplus_{k=0}^{n} \mathrm{Val}_k\,.$$

Proof. (1) Uniqueness: Let $\phi = \phi_0 + \ldots + \phi_n = \phi_0' + \ldots + \phi_n'$. For $K \in \mathcal{K}(V)$ and $t \geq 0$ we obtain

$$\phi(tK) = \phi_0(K) + \ldots + t^n\phi_n(K) = \phi_0'(K) + \ldots + t^n\phi_n'(K).$$

Comparing coefficients, we obtain $\phi_k(K) = \phi_k'(K)$. Since K is arbitrary, $\phi_k = \phi_k'$.

(2) Existence: Recall that a polytope is the convex hull of finitely many points. By the Weyl-Minkowski theorem, a polytope is the same thing as a bounded polyhedron, i.e. a bounded intersection of finitely many half-spaces. From this description it is clear that the intersection of two polytopes is always a polytope. Each polytope can be triangulated. The space of polytopes in V is denoted by $\mathcal{P}(V)$.

Let ψ be a *simple* valuation on polytopes, i.e. $\psi : \mathcal{P}(V) \to \mathbb{C}$; $\psi(P) = 0$ if $\dim P < n$ and

$$\psi(P_1 \cup P_2) = \psi(P_1) + \psi(P_2) - \psi(P_1 \cap P_2)$$

whenever $P_1, P_2, P_1 \cup P_2 \in \mathcal{P}(V)$.

Claim: For $P \in \mathcal{P}(V)$, there are complex numbers $\psi_0(P), \ldots, \psi_n(P)$ such that

$$\psi(tP) = \sum_{k=0}^{n} t^k\psi_k(P), \quad \forall t = 1, 2, 3, \ldots.$$

Proof of the claim: Since ψ is simple and P can be triangulated, it suffices to show the claim in the case where $P = \Delta$ is a simplex (i.e. the convex hull of $n+1$ points). We can fix our system of coordinates in such a way that

$$P = \Delta = \{(x_1, \ldots, x_n) | 1 \geq x_1 \geq x_2 \geq \ldots \geq x_n \geq 0\}.$$

For a permutation σ, we set

$$\Delta_\sigma = \{(x_1, \ldots, x_n) | 1 \geq x_{\sigma(1)} \geq x_{\sigma(2)} \geq \ldots \geq x_{\sigma(n)} \geq 0\}.$$

and for $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, we set $\Delta_\sigma^a = \Delta_\sigma + a$.

The Δ_σ^a recover $V = \mathbb{R}^n$: for $x = (x_1, \ldots, x_n)$ there is a permutation σ such that $\{x_{\sigma(1)}\} \geq \ldots \geq \{x_{\sigma(n)}\}$ ($\{x\}$ is the fractional part of x and $\lfloor x \rfloor$ the integer part). With $a_i := \lfloor x_i \rfloor$, we have $x \in \Delta_\sigma^a$. The intersection $\Delta_\sigma^a \cap \Delta_{\sigma'}^{a'}$ is of dimension $< n$ if $(\sigma, a) \neq (\sigma', a')$.

We have for $t = 1, 2, \ldots$

$$t\Delta = \{(x_1, \ldots, x_n) | t \geq x_1 \geq \ldots \geq x_n \geq 0\}.$$

Hence $t\Delta$ is the union of those Δ_σ^a with $\Delta_\sigma^a \subseteq t\Delta$.

Fix σ. Then $\Delta_\sigma^a \subseteq t\Delta \iff t \geq x_1 + a_1 \geq x_2 + a_2 \ldots \geq x_n + a_n \geq 0$ for all $(x_1, \ldots, x_n) \in \Delta_\sigma$. This is the case if and only if

$$t - 1 \geq a_1 ?_1 a_2 ?_2 \ldots ?_{n-1} a_n \geq 0$$

where

$$?_i = \begin{cases} \geq & \text{if } i \text{ appears before } i+1 \text{ in } \sigma(1), \ldots, \sigma(n) \\ > & \text{else.} \end{cases}$$

Let $k = k_\sigma$ be the number of signs $\geq$. By changing variables, we have

$$\begin{aligned} \#\{(a_1, \ldots, a_n) | t-1 \geq a_1 ?_1 a_2 ?_2 \ldots ?_{n-1} a_n \geq 0\} \\ = \#\{(a_1', \ldots, a_n') | t+k-1 \geq a_1' > a_2' > a_3' \ldots > a_n' \geq 0\} \\ = \binom{t+k}{n}. \end{aligned}$$

We deduce that

$$\begin{aligned} \psi(t\Delta) = \psi\left(\bigcup_{\Delta_\sigma^a \subseteq t\Delta} \Delta_\sigma^a\right) = \sum_{\sigma, a | \Delta_\sigma^a \subseteq t\Delta} \psi(\Delta_\sigma^a) = \sum_\sigma \#\{a : \Delta_\sigma^a \subseteq t\Delta\} \psi(\Delta_\sigma) \\ = \sum_\sigma \binom{t+k_\sigma}{n} \psi(\Delta_\sigma). \end{aligned}$$

This is a polynomial in t of degree $\leq n$, which shows the claim.

(3) Claim: $\phi(tP)$ is a polynomial in t of degree $\leq n$ for $t = 1, 2, 3, \ldots$ (recall that in contrast to the last case, ϕ is not assumed to be simple).

We use induction on $n = \dim V$. The case $n = 0$ is trivial.

For $n \geq 1$, we define a simple valuation ψ on polytopes by using a Euclidean structure and setting

$$\beta(P,F) := \lim_{r \to 0} \frac{\mathrm{vol}_n(P \cap B(x,r))}{\mathrm{vol}_n B(x,r)}, \quad x \in \operatorname{relint} F$$
$$\psi(P) := \sum_F (-1)^{n-\dim F} \beta(P,F) \phi(F).$$

In the first equation, F is a face of P, in the second equation, the sum is over all faces of P.

Then ψ is a simple valuation on polytopes. From $\beta(tP,tF) = \beta(P,F)$ we obtain

$$\underbrace{\psi(tP)}_{\text{polynomial in } t} = \sum_F (-1)^{n-\dim F} \beta(P,F) \underbrace{\phi(tF)}_{\text{polynomial in } t \text{ if } \dim F<n \text{ by induction hyp.}}$$

The only face of dimension n is P itself, hence $\phi(tP)$ is a polynomial of degree $\leq n$, i.e.

$$\phi(tP) = \sum_{k=0}^n t^k \phi_k(P), \quad \forall t \in \mathbb{N}_+, \tag{1}$$

where $\phi_k(P)$ is some constant depending on P and k.

(4) Taking $n+1$ pairwise different t's and using the fact that the Vandermonde matrix is non singular, we can write $\phi_k(P)$ as a linear combination of the numbers $\phi(t_i P)$. However, $P \mapsto \phi(t_i P)$ is a valuation, hence ϕ_k is also a valuation.

(5) Let $q \in \mathbb{Q}_+$. For $t \in \mathbb{N}_+$ such that $tq \in \mathbb{N}_+$, we have

$$\phi(tqP) = \phi(t(qP)) = \sum_{k=0}^n t^k \phi_k(qP)$$
$$\phi(tqP) = \sum_{k=0}^n (tq)^k \phi_k(P).$$

But there are infinitely many such t, hence we obtain $\phi_k(qP) = q^k \phi_k(P)$ for all polytopes P.

(6) Let $t \in \mathbb{Q}_+$, then (replacing P by tP in (1))

$$\phi(tP) = \sum_{k=0}^n \phi_k(tP) = \sum_{k=0}^n t^k \phi_k(P).$$

By continuity, this equation holds for all $t > 0$.

(7) Fix numbers $0 < t_0 < \ldots < t_n$. We obtain

$$\phi(t_i P) = \sum_{k=0}^n t_i^k \phi_k(P).$$

Since the Vandermonde matrix is non-singular, there exist constants c_{kj} (which are independent of P) such that

$$\phi_k(P) = \sum_{j=0}^{n} c_{kj}\phi(t_j P)$$

for all polytopes P. We define ϕ_k on $\mathcal{K}(V)$ by this equation, i.e.

$$\phi_k(K) := \sum_{j=0}^{n} c_{kj}\phi(t_j K).$$

Since ϕ is continuous and translations invariant, we also have, $\phi_k \in$ Val. Polytopes being dense in $\mathcal{K}(V)$, we get by continuity

$$\phi(tK) = \sum_{k=0}^{n} t^k \phi_k(K).$$

In particular, $(t = 1)$ $\phi = \phi_0 + \ldots + \phi_n$ and $\phi_k \in \mathrm{Val}_k$.

□

Corollary 1.2. *Let $C \in \mathcal{K}(V)$ be a fixed convex body with non-empty interior. The space* Val *becomes a Banach space under the norm*

$$\|\phi\| = \sup\{|\phi(K)| : K \subseteq C\}.$$

Moreover, a different choice of C yields an equivalent norm.

Proof. It is not difficult to show that $\|\cdot\|$ defines a norm on Val and that Val is complete with respect to the induced topology. Let $C_1, C_2 \in \mathcal{K}(V)$ be convex bodies containing the origin in their interior. We may assume that

$$C_1 \subseteq C_2 \subseteq aC_1 \tag{2}$$

for some constant $a > 0$. Consider the two norms on Val defined by

$$\|\phi\|_j = \sup\{|\phi(K)| : K \subseteq C_j\}, \qquad j = 1, 2.$$

Clearly, by (2), we have $\|\phi\|_1 \leq \|\phi\|_2$ for every $\phi \in$ Val.

It remains to show that there is a constant $c > 0$ such that $\|\phi\|_2 \leq c\,\|\phi\|_1$ for every $\phi \in$ Val. In order to see this, consider for fixed $\phi \in$ Val and $K \in \mathcal{K}(V)$ the function $p_K(t) := \phi(tK), t \geq 0$. By McMullen's theorem, p_K is a polynomial of degree at most n. Clearly, $\|\phi\|_j = \sup\{|p_K(1)| : K \subseteq C_j\}, j = 1, 2$.

Since on the space of polynomials of degree at most n all norms are equivalent, there exists a constant $c > 0$ (depending on n and a only) such that for every polynomial q of degree at most n we have

$$|q(1)| \leq c \sup\{|q(t)| : 0 \leq t \leq 1/a\}.$$

Consequently, using (2) again, we obtain

$$\|\phi\|_2 \leq c \sup\{|\phi(tK)| : K \subseteq C_2, 0 \leq t \leq 1/a\} \leq c\,\|\phi\|_1.$$

□

A valuation $\phi \in \mathrm{Val}$ is called *even* if $\phi(-K) = \phi(K)$ and odd if $\phi(-K) = -\phi(K)$. If ϕ is arbitrary, then $\phi = \phi^+ + \phi^-$ with $\phi^+(K) := \frac{1}{2}(\phi(K) + \phi(-K))$ even and $\phi^-(K) := \frac{1}{2}(\phi(K) - \phi(-K))$ odd. The McMullen decomposition can thus be refined as

$$\mathrm{Val} = \bigoplus_{\substack{k=0,\ldots,n \\ \epsilon=\pm}} \mathrm{Val}_k^\epsilon .$$

2. Lecture 2: The Klain embedding

The aim of this lecture is a description of the space of *even* valuations of a given degree. As a byproduct, we will obtain a proof of Hadwiger's theorem.

Theorem 2.1 (Volume characterization, Klain [11]). *Let μ be a continuous valuation which is*

(1) *translation invariant,*
(2) *simple, i.e. $\mu(K) = 0$ if $\dim K < n = \dim V$,*
(3) *even, i.e. $\mu(-K) = \mu(K)$.*

Then there exists $c \in \mathbb{C}$ such that $\mu = c\mu_n$.

Proof. (1) We fix a basis $\mathcal{B}$ of V, which allows us to identify V and $\mathbb{R}^n$. Let $c := \mu([0,1]^n)$ and $\tilde{\mu} := \mu - c\mu_n$. Then $\tilde{\mu}$ satisfies the same properties as μ and moreover $\tilde{\mu}([0,1]^n) = 0$. It is enough to show that $\tilde{\mu} = 0$.

(2) We use induction on n in order to show that a simple, continuous, translation invariant, even valuation μ with $\mu([0,1]^n) = 0$ vanishes. The induction start $n = 1$ is trivial. Let $n > 1$.

Since $\mu([0,1]^n) = 0$, we deduce that $\mu([0,1/k]^n) = 0$ for all $k \in \mathbb{N}$: indeed $[0,1]^n$ is the union of translates of $[0,1/k]^n$ and the pairwise intersections are of smaller dimensions.

A box P with rational side lengths and faces which are parallel to the coordinate axes can be cut into a finite number of such cubes. Hence $\mu(P) = 0$. By continuity of μ, $\mu(P) = 0$ also if the side lengths are not rational.

(3) Suppose $n = 2$. A parallelogram can be cut into a finite number of pieces such that their translations form a parallelogram whose sides are parallel to the coordinate axes.

(4)

$$SO(n, \mathcal{B}) := \{g \in SO(n) | g \text{ fixes at least } n-2 \text{ basis vectors of } \mathcal{B}\}$$

Then each element $g \in SO(n)$ is the product of a finite number of elements in $SO(n, \mathcal{B})$:

If $ge_1 = e_1$, then g fixes the space V' spanned by $e_2, \ldots, e_n$. Let $\mathcal{B}' := \{e_2, \ldots, e_n\}$. $g|_{V'} \in SO(n-1) \implies g|_{V'} = g_1 \cdot \ldots \cdot g_k$ with $g_1, \ldots, g_k \in SO(n-1, \mathcal{B}')$ by induction. Each g_i can be extended to an element $\tilde{g}_i \in SO(n, \mathcal{B})$ by setting $\tilde{g}_i e_1 := e_1$. It follows that $g = \tilde{g}_1 \cdot \ldots \cdot \tilde{g}_k$.

If $v := ge_1 \neq e_1$, set $v' := \frac{v - \langle v, e_1 \rangle e_1}{\|v - \langle v, e_1 \rangle e_1\|} \in span\{e_2, \ldots, e_n\}, \|v'\| = 1$. There exists $\psi \in SO(n)$ such that $\psi(e_1) = e_1, \psi v' = e_2$. Let ξ be the rotation which fixes $e_3, \ldots, e_n$ and which sends $\psi v \in span\{e_1, e_2\}$ to e_1. Then $\xi \in SO(n, \mathcal{B})$ and $\eta := \xi \circ \psi \circ g$ satisfies $\eta e_1 = \xi \psi g e_1 = \xi \psi v = e_1$. By the case which we already studied, there exists $\psi_1, \ldots, \psi_i, \eta_1, \ldots, \eta_j \in SO(n, \mathcal{B})$ with $\psi = \psi_1 \cdot \ldots \cdot \psi_i$ and $\eta = \eta_1 \cdot \ldots \cdot \eta_j$. Hence

$$g = \psi^{-1} \xi^{-1} \eta = \underbrace{\psi_i^{-1} \cdot \ldots \cdot \psi_1^{-1} \xi^{-1} \eta_1 \cdot \ldots \cdot \eta_j}_{\text{all in } SO(n, \mathcal{B})}.$$

(5) For $g \in SO(n, \mathcal{B})$, a box P can be de- and recomposed into the box gP. Indeed, g fixes $n-2$ coordinate axes and we can apply Step 3. By Step 4, this is even true for all $g \in SO(n)$. It follows by Step 2 that $\mu(P) = 0$ for all boxes.

(6) Let W be a hyperplane and $\mathbb{R}^n = W \times \mathbb{R}$ the orthogonal decomposition. We define a valuation τ on W by

$$\tau(K) := \mu(K \times [0, 1]).$$

Then τ is a continuous, translation invariant simple valuation which vanishes on boxes in W. By induction $\tau = 0$. Since μ is simple and continuous, we obtain $\mu(K \times [a, b]) = 0$ for all $a \leq b$. Hence μ vanishes on all right cylinders with convex base.

(7) A cylinder which is not a right one (i.e. a prism) can be cut by an affine hyperplane into two pieces which can be rearranged to get a right cylinder (actually this is true if the height is large enough with respect to the angle, which by additivity we can always achieve). Hence μ vanishes on all cylinders.

(8) Let P be a convex polytope with facets (i.e. $n-1$-dimensional faces) $P_1, \ldots, P_m$. Let u_i be the normal vector to P_i. Fix $v \in \mathbb{R}^n \setminus \{0\}$. For each u, we either have $\langle u_i, v \rangle > 0$ or $\langle u_i, v \rangle \leq 0$. Without loss of generality, let $P_1, \ldots, P_j$ the faces with $\langle u_i, v \rangle > 0$.

Then $P + [0, v]$ is the union of P and the cylinders $P_i + [0, v], 1 \leq i \leq j$. Pairwise intersections are of smaller dimension. Since μ is simple,

$$\mu(P + [0, v]) = \mu(P) + \sum_{i=1}^{j} \underbrace{\mu(P_i + [0, v])}_{=0, \text{ by Step 7}}.$$

(9) A *zonotope* Z is a finite Minkowski sum of intervals. A *zonoid* is the limit (in the Hausdorff topology) of a sequence of zonotopes. By Step 8,

$$\mu(P + Z) = \mu(P)$$

for each polytope P. In particular, $\mu(Z) = 0$. By continuity of μ,

$$\mu(Y) = 0, \mu(K + Y) = \mu(K)$$

for each zonoid Y and $K \in \mathcal{K}(V)$. Here we use that polytopes are dense in $\mathcal{K}(V)$.

(10) Let $K \in \mathcal{K}_o(V)$ (i.e. origin symmetric with non empty interior) with smooth support function. Then for large $r > 0$, $K + rB$ is a zonoid. The proof of this fact uses the so called cosine function. Since rB is a zonoid itself, we obtain $\mu(K) = \mu(K + rB) = 0$.

(11) A general convex body $K \in \mathcal{K}_o(V)$ can be approximated by a sequence $K_1, \ldots,$ of origin symmetric convex bodies with smooth support function. By continuity, $\mu(K) = \lim_{i\to\infty} \mu(K_i) = 0$. Hence μ vanishes on $\mathcal{K}_o(V)$.

(12) Let $\Delta = [0, u_1, \ldots, u_n]$ be a simplex, $v := u_1 + \ldots + u_n$, $P := [0, u_1] + [0, u_2] + \cdots + [0, u_n]$. Let E_1 be the affine hyperplane going through $u_1, \ldots, u_n$. Let E_2 be the affine hyperplane going through $v - u_1, \ldots, v - u_n$. Let P^* be the intersection of P with the strip between E_1 et E_2.

Then
$$P = \Delta \cup P^* \cup (-\Delta + v).$$
Since P^* is symmetric with respect to $v/2$, we have $\mu(P^*) = 0$. Since μ is simple, translation invariant and even,
$$0 = \mu(P) = \mu(\Delta) + \mu(P^*) + \mu(-\Delta + v) = 2\mu(\Delta).$$
Hence μ vanishes on simplices.

(13) Let P be a polytope. Then P can be triangulated: $P = \Delta_1 \cup \ldots \cup \Delta_m$. Hence
$$\mu(P) = \sum_{i=1}^m \mu(\Delta_i) = 0.$$

(14) $K \in \mathcal{K}(V)$ can be approximated by polytopes $P_1, P_2, \ldots$. Hence
$$\mu(K) = \lim_{i\to\infty} \mu(P_i) = 0.$$
We thus finally have $\mu = 0$.

□

Lemma 2.2 (Sah)**.** *Let Δ be a simplex of dimension n. Then there exist polytopes $P_1, \ldots, P_m$ such that*
$$\Delta = P_1 \cup \ldots \cup P_m, \dim P_i \cap P_j < n$$
and such that P_i is symmetric with respect to some affine hyperplane.

Proof. Let $x_0, \ldots, x_n$ be the vertices of Δ, Δ_i the facet opposite to x_i, z the center of the inscribed circle in Δ, z_i the orthogonal projection of z on Δ_i. We set
$$A_{i,j} := [z, z_i, z_j, \Delta_i \cap \Delta_j].$$
Then
$$\Delta = \bigcup_{0 \le i < j \le n} A_{i,j}, \quad \dim A_{i,j} \cap A_{i',j'} < n.$$

$A_{i,j}$ is symmetric with respect to the affine hyperplane passing through z and containing $\Delta_i \cap \Delta_j$. $\square$

Theorem 2.3 (Hadwiger, 1957). *Let μ be a valuation on a Euclidean vector space V of dimension n such that*

(1) *μ is continuous,*
(2) *μ is translation invariant,*
(3) *μ is rotation invariant, i.e. invariant under the group $SO(n)$.*

Then there exist constants $c_0, \ldots, c_n \in \mathbb{C}$ such that

$$\mu = \sum_{k=0}^{n} c_k \mu_k.$$

Proof. Let $W \subseteq V$ be an affine hyperplane. $\mu|_W$ is a continuous, translation and rotation invariant valuation on W. By induction,

$$\mu|_W = \sum_{k=0}^{n-1} c_k \mu_k|_W.$$

Let

$$\tilde{\mu} := \mu - \sum_{k=0}^{n-1} c_k \mu_k.$$

Then $\tilde{\mu}|_W = 0$. Since $\tilde{\mu}$ is invariant by rotations, it is simple.

If $n \equiv 0 \mod 2$, then $-Id \in SO(n)$ and $\tilde{\mu}$ is even.

If $n \equiv 1 \mod 2$, let $\Delta = P_1 \cup \ldots \cup P_m$ as in Sah's lemma. Since P_i is symmetric with respect to some affine hyperplane, $-P_i$ and P_i agree up to a rotation. Hence

$$\tilde{\mu}(-\Delta) = \sum_{i=1}^{m} \tilde{\mu}(-P_i) = \sum_{i=1}^{m} \tilde{\mu}(P_i) = \tilde{\mu}(\Delta).$$

Using a triangulation and a continuity argument as before, we obtain that $\tilde{\mu}$ is even.

In both cases, by Klain's theorem, $\tilde{\mu} = c_n \mu_n$ with $c_n \in \mathbb{C}$. Then

$$\mu = \sum_{k=0}^{n} c_k \mu_k.$$

$\square$

Let $\phi \in \mathrm{Val}_+^k$. For $E \in \mathrm{Gr}_k(V)$, the restriction $\phi|_E$ is continuous, translation invariant. We claim that $\phi|_E$ is simple. Otherwise, take a minimal subspace $F \subseteq E$ such that $\phi|_F \neq 0$. Then $\phi|_F$ is simple. By Klain's theorem, $\phi|_F = c \,\mathrm{vol}_F$. For $K \in \mathcal{K}(F)$ we have $\phi(tK) = t^k \phi(K)$ since ϕ is of degree k. On the other hand, $\phi(tK) = c \,\mathrm{vol}_F(tK) = t^{dim F} \phi(K)$. But $\dim F < k$, which implies the contradiction $\phi|_F = 0$.

By Klain's theorem, $\phi|_E = c(E) \,\mathrm{vol}_E$ with some constant $c(E) \in \mathbb{C}$.

Definition 1. *The map* $\mathrm{Kl}_\phi : \mathrm{Gr}_k(V) \to \mathbb{C}, E \mapsto c(E)$ *is called Klain function. It is continuous on* $\mathrm{Gr}_k(V)$*.*

Proposition 2.4 (Klain [12])**.** *The Klain map* $\mathrm{Kl} : \mathrm{Val}_k^+ \to C(\mathrm{Gr}_k(V)), \phi \mapsto \mathrm{Kl}_\phi$ *is injective. Hence* Val_k^+ *can be considered as a subspace of* $C(\mathrm{Gr}_k(V))$*.*

Proof. Clearly Kl is linear. We have to show that $\mathrm{Kl}_\phi = 0$ implies $\phi = 0$. Suppose that $\phi \neq 0$. Let F be a subspace of minimal dimension with $\phi|_F \neq 0$. Then $\phi|_F$ is simple. By the same argument as above, $\dim F = k$ and $\phi|_F = c \operatorname{vol}_F, c \neq 0$. On the other hand, $c = \mathrm{Kl}_\phi(F) = 0$, contradiction. $\square$

3. Lecture 3: Algebraic structures on the space of valuations

A *representation* of a Lie group G (which will be the general linear group $GL(n)$ in the following) on a Banach space X (which will be Val) is a continuous map $G \times X \to X, (g, x) \mapsto gx$ such that $x \mapsto gx$ is linear, $1x = x$ and $g(hx) = (gh)x$. A linear subspace $Y \subseteq X$ is called *invariant* if $gy \in Y$ for all $g \in G, y \in Y$. There are two obvious examples of invariant subspaces, namely $Y = \{0\}$ and $Y = X$. The representation is called *irreducible* if there are no other invariant and *dense* subspaces in X.

Remark:

(1) If $\dim X < \infty$, every linear subspace is closed.
(2) If the representation is irreducible, every invariant linear subspace is either trivial or dense in X. This follows since the closure $\bar{Y}$ is also invariant, hence $\bar{Y} = X$.

In our situation $X := (\mathrm{Val}, \|\cdot\|)$ and $G = GL(n)$. If $g \in GL(n)$ and $K \in \mathcal{K}(V)$, then gK is also compact and convex. For $\phi \in \mathrm{Val}$, we define $g\phi \in \mathrm{Val}$ by

$$g\phi(K) := \phi(g^{-1}K).$$

This is a representation of $GL(n)$ on Val. The subspaces $\mathrm{Val}_k^\pm$ are invariant: if $\phi \in \mathrm{Val}_k$, then $g\phi(tK) = \phi(g^{-1}tK) = \phi(tg^{-1}K) = t^k\phi(g^{-1}K) = t^k g\phi(K)$, hence $g\phi \in \mathrm{Val}_k$.

The next theorem is a milestone of modern integral geometry.

Theorem 3.1 (Alesker [1])**.** *The representation of* $GL(n)$ *on* Val_k^ϵ *is irreducible for each* $k = 0, \ldots, n$ *and each* $\epsilon = \pm$*.*

The proof is far beyond the scope of these lecture notes.

Corollary 3.2 (McMullen's conjecture [1])**.** *Linear combinations of valuations of the type* $K \mapsto \operatorname{vol}(K + A), A \in \mathcal{K}(V)$ *are dense in* Val*.*

Proof. Let Y be the space of all linear combinations of valuations of the form $K \mapsto \operatorname{vol}(K + A), A \in \mathcal{K}(V)$. We have to show that $\bar{Y} = \mathrm{Val}$. Fix $k = 0, \ldots, n$ and $\epsilon = \pm$. Then $Y \cap \mathrm{Val}_k^\epsilon$ is a linear subspace of Val_k^ϵ. This space is invariant under $GL(n)$, since Y and Val_k^ϵ are invariant. For Y: if

$\phi(K)=\sum_{i=1}^m c_i \operatorname{vol}(K+A_i) \in Y$, then

$$g\phi(K)=\sum_{i=1}^m c_i \operatorname{vol}(g^{-1}K+A_i)=\sum_{i=1}^m c_i \operatorname{vol}(K+gA_i) \det g^{-1}=\sum_{i=1}^m \tilde c_i \operatorname{vol}(K+\tilde A_i) \in Y$$

where $\tilde c_i := c_i \det g^{-1}$ and $\tilde A_i := gA_i$.

Claim: $Y \cap \operatorname{Val}_k^\epsilon \neq \{0\}$. We show this only for $\epsilon=+$. It is enough to construct one element in $Y \cap \operatorname{Val}_k^\epsilon$. Fix a Euclidean structure on V. For $r>0$, $\phi_r(K):=\operatorname{vol}(K+rB)$ is in Y. We have $\phi_r=\sum_{k=0}^n \omega_k \mu_{n-k} r^k$ by Steiner's formula. Let us fix $0<r_0<\ldots<r_n$. Then

$$\phi_{r_i}=\sum_{k=0}^n \omega_k \mu_{n-k} r_i^k, i=0,\ldots,n$$

is a system of linear equations on the μ_{n-k} which we can solve:

$$\mu_{n-k}=\sum_{i=0}^n c_{k_i}\phi_{r_i} \in Y \cap \operatorname{Val}_{n-k}^+ .$$

□

Definition 2. *A valuation $\phi \in \operatorname{Val}$ is called smooth if the map $GL(V) \to \operatorname{Val}, g \mapsto g\phi$ is smooth (this is a map from a smooth manifold to a Banach space). The space of smooth valuation is denoted by* Val^{sm}.

Example: intrinsic volumes are smooth; valuations of the type $K \mapsto \operatorname{vol}(K+A)$ are smooth if A is strictly convex and smooth. Smooth valuations are dense in the space of all valuations.

Theorem 3.3 (Alesker, [3])**.** *There is a unique product structure on* Val^{sm} *such that if $\phi_i(K)=\operatorname{vol}(K+A_i)$ for smooth and strictly convex A_i, then*

$$\phi_1 \cdot \phi_2(K)=\operatorname{vol}(\Delta K + A_1 \times A_2),$$

where $\Delta : V \to V \times V$ is the diagonal embedding.

The proof is rather difficult. Uniqueness follows from the fact that linear combinations of valuations of the form $K \mapsto \operatorname{vol}(K+A)$ are dense in Val^{sm}.

The product is compatible with $GL(n)$ in the following sense: for $g \in GL(n), \phi_1, \phi_2 \in \operatorname{Val}^{sm}$ we have $g(\phi_1 \cdot \phi_2)=g\phi_1 \cdot g\phi_2$. In particular, if ϕ_1, ϕ_2 are invariant under some subgroup of $GL(n)$, then also the product is invariant. From Hadwiger's theorem it follows that for intrinsic volumes we have $\mu_i \cdot \mu_j = c_{i,j}\mu_{i+j}$. We will compute the constants later.

Theorem 3.4 (B'-Fu, [8])**.** *There is a unique convolution product on* Val^{sm} *such that if $\phi_i(K)=\operatorname{vol}(K+A_i)$ for smooth and strictly convex A_i, then*

$$\phi_1 \cdot \phi_2(K)=\operatorname{vol}(K+A_1+A_2).$$

Again, uniqueness is clear.

Theorem 3.5 (Alesker [2, 4])**.** *There is a Fourier type transform* $\hat{}: \mathrm{Val}^{sm} \to \mathrm{Val}^{sm}$ *such that*

$$\hat{\phi}_1 * \hat{\phi}_2 = \widehat{\phi_1 \cdot \phi_2}.$$

In the even case, it can be characterized as follows. If $\phi \in \mathrm{Val}_k^{sm,+}$*, then* $\hat{\phi} \in \mathrm{Val}_{n-k}^{sm,+}$ *satisfies*

$$\mathrm{Kl}_{\hat{\phi}}(E) = \mathrm{Kl}_{\phi}(E^{\perp}), \quad E \in \mathrm{Gr}_{n-k}(V).$$

In the end of this lecture, we want to compute product and convolution of intrinsic volumes.

First of all, $\mu_k \in \mathrm{Val}_k^{sm,+}$ is characterized by its Klain function $\mathrm{Kl}_{\mu_k}(E) = 1$ for all $E \in \mathrm{Gr}_k(V)$. Then by definition of the Fourier transform (in the even case) we get

$$\hat{\mu}_k = \mu_{n-k}.$$

Next we compute the convolution. Let ω_n be the volume of the n-dimensional unit ball, and define the *flag coefficient*

$$\left[\begin{array}{c} n \\ i \end{array} \right] := \frac{\omega_n}{\omega_i \omega_{n-i}} \binom{n}{i}.$$

Recall Steiner's formula:

$$\mathrm{vol}(K + rB) = \sum_{i=0}^{n} \mu_{n-i}(K) \omega_i r^i.$$

Now fix r and s and define $\phi_1(K) := \mathrm{vol}(K + rB), \phi_2(K) := \mathrm{vol}(K + sB)$. Then

$$\phi_1 * \phi_2(K) = \mathrm{vol}(K + rB + sB) = \mathrm{vol}(K + (r+s)B) = \sum_{k=0}^{n} \mu_{n-k}(K) \omega_k (r+s)^k,$$

hence

$$\phi_1 * \phi_2 = \sum_{i,j=0}^{n} \mu_{n-i-j} \omega_{i+j} \binom{i+j}{i} r^i s^j.$$

On the other hand, since $\phi_1 = \sum_{i=0}^{n} \mu_{n-i} \omega_i r^i$ and $\phi_2 = \sum_{i=0}^{n} \mu_{n-i} \omega_i s^i$, we obtain

$$\phi_1 * \phi_2 = \sum_{i,j=0}^{n} \mu_{n-i} * \mu_{n-j} \omega_i \omega_j r^i s^j.$$

Now compare the coefficient of $r^i s^j$ in these equations:

$$\mu_{n-i-j} \omega_{i+j} \binom{i+j}{i} = \mu_{n-i} * \mu_{n-j} \omega_i \omega_j.$$

Writing i instead of $n - i$ and j instead of $n - j$, we obtain

$$\mu_i * \mu_j = \left[\begin{array}{c} 2n - i - j \\ n - i \end{array} \right] \mu_{i+j-n}. \tag{3}$$

Taking Fourier transform on both sides yields

$$\mu_i \cdot \mu_j = \begin{bmatrix} i+j \\ i \end{bmatrix} \mu_{i+j}. \tag{4}$$

4. Lecture 4: Integral geometry of $SO(n)$ and other groups

For a subgroup G of $GL(n)$, let Val^G denote the subspace of G-invariant valuations.

Let $G := SO(n)$ and $V := \mathbb{R}^n$. We want to determine the integral

$$\int_G \mu_i(K + gL)dg.$$

For each fixed body L, the left hand side of this formula is a valuation in K. It is easy to prove that this valuation belongs to Val^G, hence (by Hadwiger's theorem) it may be written in the form $\sum_{k=0}^n d_k(L)\mu_k(K)$. Next, fixing K, one easily gets that d_k is also an element of Val^G for each fixed k, hence $d_k(L) = \sum_{l=0}^n d_{kl}\mu_l(L)$ with complex numbers d_{kl}. We thus know that

$$\int_G \mu_i(K + gL)dg = \sum_{k,l=0}^n d^i_{kl}\mu_k(K)\mu_l(L)$$

for some fixed constants d^i_{kl}. There is a nice trick to determine these constants, which is called the *template method*. We plug in on both sides of the equation special convex bodies K and L for which we may compute the integral on the left hand side and the intrinsic volumes on the right hand side to obtain a system of linear equations on the d^i_{kl}. Solving this system yields the d^i_{kl}. More precisely, let us take $K = rB, L = sB$ (where B is as always the unit ball). Then we obtain

$$(r+s)^i \binom{n}{i} \frac{\omega_n}{\omega_{n-i}} = \sum_{k,l} d^i_{k,l} r^k \binom{n}{k} \frac{\omega_n}{\omega_{n-k}} s^l \binom{n}{l} \frac{\omega_n}{\omega_{n-l}}.$$

Comparing the coefficients of $r^j s^{i-j}$ on both sides gives us

$$\binom{i}{j}\binom{n}{i}\frac{\omega_n}{\omega_{n-i}} = d^i_{j,i-j}\binom{n}{j}\frac{\omega_n}{\omega_{n-j}}\binom{n}{i-j}\frac{\omega_n}{\omega_{n-i+j}}.$$

We thus get

$$\int_{\mathrm{SO}(V)} \mu_i(K+gL)dg = \begin{bmatrix} 2n-i \\ n-i \end{bmatrix} \sum_{k+l=i} \begin{bmatrix} 2n-i \\ n-k \end{bmatrix}^{-1} \mu_k(K)\mu_l(L). \tag{5}$$

The same argument works also for the usual kinematic formula. For this, we denote by $\overline{\mathrm{SO}(V)}$ the Euclidean motion group and normalize its Haar measure in such a way that the measure of elements $\bar{g}$ with $\bar{g}(0) \in K$ equals $\operatorname{vol} K$ for each compact set K. Then we obtain

$$\int_{\overline{\mathrm{SO}(V)}} \mu_i(K \cap \bar{g}L)d\bar{g} = \begin{bmatrix} n+i \\ i \end{bmatrix} \sum_{k+l=n+i} \begin{bmatrix} n+i \\ k \end{bmatrix}^{-1} \mu_k(K)\mu_l(L). \tag{6}$$

Theorem 4.1 (Alesker 2007,). *A compact subgroup G of $\mathrm{SO}(n), n \geq 2$ satisfies $\dim \mathrm{Val}^G < \infty$ if and only if G acts transitively on the unit sphere. In this case, every G-invariant, translation invariant and continuous valuation is smooth.*

The classification of connected compact Lie groups G acting transitively on some sphere is a topological problem which was solved by Montgomery-Samelson and Borel. There are six infinite lists

$$\mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n), \mathrm{Sp}(n), \mathrm{Sp}(n) \cdot \mathrm{U}(1), \mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \tag{7}$$

and three exceptional groups

$$\mathrm{G}_2, \mathrm{Spin}(7), \mathrm{Spin}(9). \tag{8}$$

Recently, kinematic formulas for the groups $\mathrm{U}(n), \mathrm{SU}(n), \mathrm{G}_2, \mathrm{Spin}(7)$ were obtained [9, 7, 6]. The template method described above is not strong enough to yield these formulas. Instead, one has to use a more algebraic approach which we will describe now.

Let V be a Euclidean vector space and let G be a subgroup of $\mathrm{SO}(V)$ which acts transitively on the unit sphere.

It was observed by Hadwiger that $\mathrm{Val}_n = \mathbb{C} \operatorname{vol}_n$. Define a pairing

$$\mathrm{Val}^G \times \mathrm{Val}^G \to \mathbb{C}, (\phi, \psi) \mapsto \langle \phi, \psi \rangle,$$

such that $\phi \cdot \psi = \langle \phi, \psi \rangle \operatorname{vol}_n$.

Theorem 4.2 (Alesker 2003, [3]). *The Alesker-Poincare pairing is non-degenerate, i.e. the induced map*

$$\mathrm{PD}_G : \mathrm{Val}^G \to \mathrm{Val}^{G*}, \phi \mapsto \langle \phi, \cdot \rangle$$

is a bijection.

If $\phi_1, \ldots, \phi_m$ is a basis of Val^G, then by the same trick as above we obtain kinematic formulas

$$\int_{\bar{G}} \phi_i(K \cap \bar{g}L) d\bar{g} = \sum_{k,l=1}^{m} c_{k,l}^i \phi_k(K) \phi_l(L). \tag{9}$$

There is a very nice and clever way to encode these formulas in a purely algebraic way. Define the kinematic operator

$$\begin{aligned} k_G : \mathrm{Val}^G &\to \mathrm{Val}^G \otimes \mathrm{Val}^G \\ \phi_i &\mapsto \sum_{k,l=1}^{m} c_{k,l}^i \phi_k \otimes \phi_l. \end{aligned}$$

This map is in fact a *cocommutative, coassociative coproduct* on Val^G.

For instance, cocommutativity means that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Val}^G & \xrightarrow{k_G} & \mathrm{Val}^G \otimes \mathrm{Val}^G \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \iota} \\ \mathrm{Val}^G & \xrightarrow{k_G} & \mathrm{Val}^G \otimes \mathrm{Val}^G\,. \end{array}$$

Here ι is the map that interchanges the factors of $\mathrm{Val}^G \otimes \mathrm{Val}^G$.

In more concrete terms, this says that the coefficients in the kinematic formula (9) satisfy $c^i_{k,l} = c^i_{l,k}$, which expresses the symmetry of the formula (in K and L).

The coassociativity property is the commutativity of the following diagram:

$$\begin{array}{ccc} \mathrm{Val}^G & \xrightarrow{(k_G\otimes id)\circ k_G} & \mathrm{Val}^G \otimes \mathrm{Val}^G \otimes \mathrm{Val}^G \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle id\otimes id\otimes id} \\ \mathrm{Val}^G & \xrightarrow{(id\otimes k_G)\circ k_G} & \mathrm{Val}^G \otimes \mathrm{Val}^G \otimes \mathrm{Val}^G\,. \end{array}$$

This property is equivalent to the formula

$$\sum_r c^i_{r,m} c^r_{k,l} = \sum_r c^i_{r,l} c^r_{k,m},$$

and this comes just from Fubini's theorem.

Theorem 4.3 (Fundamental theorem of algebraic integral geometry, [8])**.** *Let G be a group acting transitively on the unit sphere, $m_G : \mathrm{Val}^G \otimes \mathrm{Val}^G \to \mathrm{Val}^G$ the restriction of the Alesker product to Val^G; $\mathrm{PD}_G : \mathrm{Val}^G \to \mathrm{Val}^{G*}$ the Alesker-Poincaré duality and k_G the kinematic coproduct. Then the following diagram commutes*

$$\begin{array}{ccc} \mathrm{Val}^G & \xrightarrow{k_G} & \mathrm{Val}^G \otimes \mathrm{Val}^G \\ {\scriptstyle \mathrm{PD}_G}\downarrow & & \downarrow{\scriptstyle \mathrm{PD}_G\otimes \mathrm{PD}_G} \\ \mathrm{Val}^{G*} & \xrightarrow{m_G^*} & \mathrm{Val}^{G*} \otimes \mathrm{Val}^{G*}\,. \end{array}$$

The proof of this theorem is rather elementary and uses only some algebraic tricks.

Let us illustrate this theorem in the $SO(n)$-case. Let $\mu_0, \ldots, \mu_n$ be the intrinsic volumes and $\mu_0^*, \ldots, \mu_n^*$ be the dual basis, i.e. $\mu_i^*(\mu_j) = \delta_{ij}$. Then

$$\langle \mathrm{PD}(\mu_i), \mu_j \rangle = (\mu_i \cdot \mu_j)_n = \begin{cases} 0 & j \neq n-i \\ \begin{bmatrix} n \\ i \end{bmatrix} & i = n-j \end{cases}$$

Hence $\mathrm{PD}(\mu_i) = \begin{bmatrix} n \\ i \end{bmatrix} \mu^*_{n-i}$. Next, we compute

$$\begin{aligned}
\langle m_G^*(\mathrm{PD}(\mu_i)), \mu_a \otimes \mu_b \rangle &= \begin{bmatrix} n \\ i \end{bmatrix} \langle \mu_{n-i}^*, \mu_a \cdot \mu_b \rangle \\
&= \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} a+b \\ a \end{bmatrix} \langle \mu_{n-i}^*, \mu_{a+b} \rangle \\
&= \begin{cases} 0 & i+a+b \neq n \\ \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} a+b \\ a \end{bmatrix} & i+a+b=n \end{cases}
\end{aligned} \tag{10}$$

On the other hand,

$$k(\mu_i) = \begin{bmatrix} n+i \\ i \end{bmatrix} \sum_{k+l=n+i} \begin{bmatrix} n+i \\ k \end{bmatrix}^{-1} \mu_k \otimes \mu_l$$

and therefore

$$\mathrm{PD} \otimes \mathrm{PD}(k(\mu_i)) = \begin{bmatrix} n+i \\ i \end{bmatrix} \sum_{k+l=n+i} \begin{bmatrix} n+i \\ k \end{bmatrix}^{-1} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n \\ l \end{bmatrix} \mu_{n-k}^* \otimes \mu_{n-l}^*.$$

Then

$$\langle \mathrm{PD} \otimes \mathrm{PD}(k(\mu_i)), \mu_a \otimes \mu_b \rangle = \begin{bmatrix} n+i \\ i \end{bmatrix} \begin{bmatrix} n+i \\ n-a \end{bmatrix}^{-1} \begin{bmatrix} n \\ a \end{bmatrix} \begin{bmatrix} n \\ b \end{bmatrix} \tag{11}$$

if $a+b+i=n$ and zero otherwise.

Now one can check that (10) and (11) give the same result, since

$$\begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} a+b \\ a \end{bmatrix} = \begin{bmatrix} n+i \\ i \end{bmatrix} \begin{bmatrix} n+i \\ n-a \end{bmatrix}^{-1} \begin{bmatrix} n \\ a \end{bmatrix} \begin{bmatrix} n \\ b \end{bmatrix}$$

if $i+a+b=n$.

Using convolution instead of product, one gets a similar theorem for additive kinematic formulas, i.e. formulas of the type (5).

The real power of the method becomes evident when looking at other groups, like $G = U(n)$. Then the algebraic approach is much better than the template method described above.

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