Recent results on stereology

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Part II

1. Computation of the flower area and the wedge volume for an interior pivotal point.

— The pivotal section is convex with smooth boundary of known parametric coordinates.

— The pivotal section is a convex polygon.

2. Uniqueness properties of the invariator. Connections with the nucleator and the surfactor. Computational implications.

3. Open questions, final discussion.

The invariator principle and its applications

Computational problems

(i). To exploit the flower formula $S(\partial Y) = 4\,\mathbb{E}A(H_t)$ we have to compute the flower area of any planar convex set.

(ii). To exploit the wedge formula $V(Y) = 2\pi\ \mathbb{E}V(W_t)$ we have to compute the mean wedge volume of any planar set.

Both problems (i), (ii) have been solved for an arbitrary planar convex *n*-gon *K* in the following two cases:

(a). Interior pivotal point $O \in K$, (Cruz-Orive LM (2011) *J. Microscopy* 243, 86–102).

(b). Exterior pivotal point $O \notin K$.

Next we concentrate on case (a).

Flower area of a planar convex set with smooth boundary

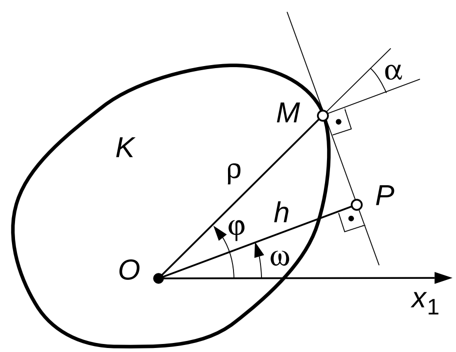
Consider a compact convex set $K \subset \mathbb{R}^2$ such that $O \in K^{\text{o}}$, with boundary ∂K of class C^2 admitting the following parametric equations,

$$\partial K = \Big\{(X, Y) \in \mathbb{R}^2 : \; X = X(t), \; X = Y(t), \; (0 \le t < 2\pi)\Big\}.$$

The parametric equations of the support functionh_K*, namely of the boundary* ∂H_K *of the flower* H_K *of* K *with respect to* O*, are the following,*

$$\left\{x(t) = -\frac{Y'(t)}{X'(t)} \cdot y(t), \;\; y(t) = \frac{X'(t)Y(t) - X(t)Y'(t)}{X'^2(t) + Y'^2(t)} \cdot X'(t), \;\; (0 \le t < 2\pi)\right\}$$

Hint of Proof.$P(x(t), y(t))$ is the intersection between the tangent to∂K at the point $M(X(t), Y(t))$ and the normal to this tangent from O. Moreover,



$$A(H_K) = \frac{1}{2}\int_0^{2\pi} h^2(\omega)\,\mathrm{d}\omega = \frac{1}{2}\int_0^{2\pi} h^2(t)\,\omega'(t)\,\mathrm{d}t = \frac{1}{2}\int_0^{2\pi} \frac{[X'(t)Y(t) - X(t)Y'(t)]^2[X'(t)Y''(t) - X''(t)Y'(t)]}{[X'^2(t) + Y'^2(t)]^2}\,\mathrm{d}t.$$

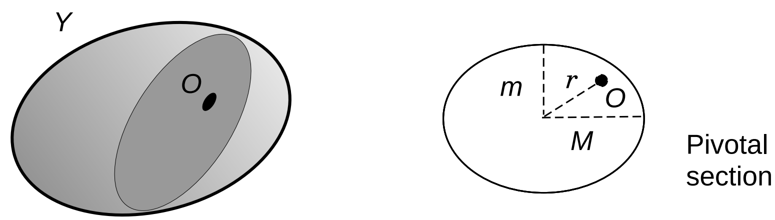
Case of an ellipse

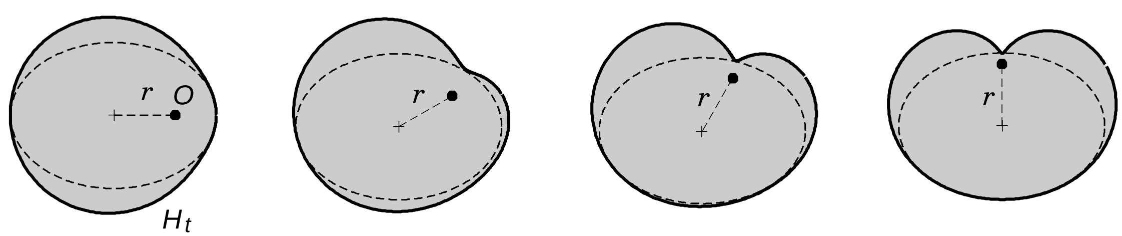
Let ∂K be an ellipse of principal semiaxes $0 < m \leq M < \infty$, and let $(x_0, y_0) \in K$ represent the rectangular coordinates of the pivotal point with respect to the ellipse centre. Shift the origin to this pivotal point. Then,

$$\partial K : \begin{cases} X(t) = M \cos t - x_0, \\ Y(t) = m \sin t - y_0, \quad (0 \leq t < 2\pi). \end{cases}$$

$$A(H_K) = \frac{\pi}{2}\left(M^2 + m^2 + r^2\right),$$

where $r^2 := x_0^2 + y_0^2$.



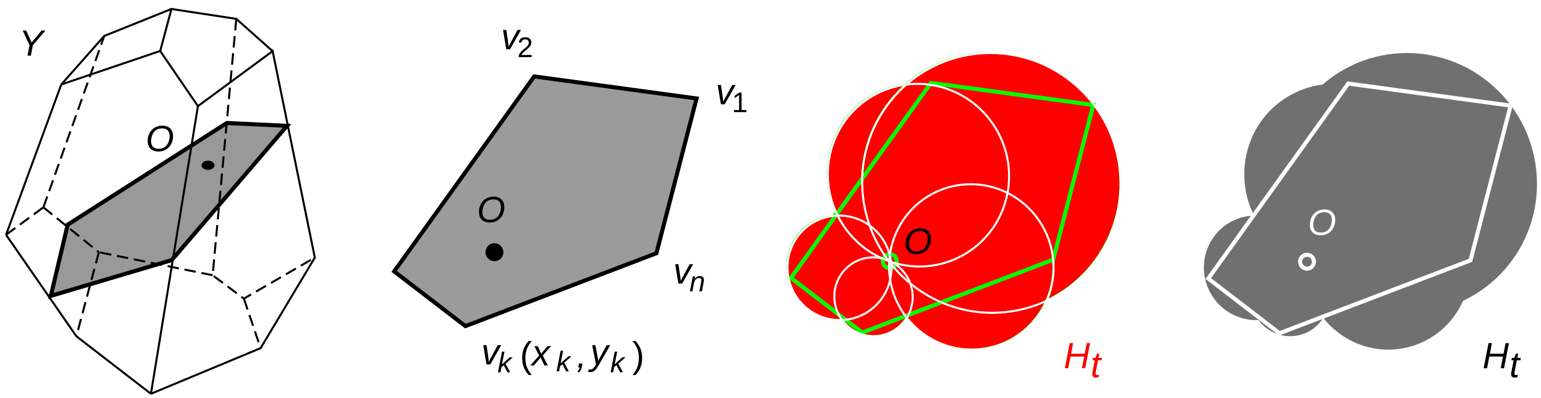


Flowers of identical areas if *r* is fixed

Flower area of a convex polygon

A pivotal section of a convex particle is a.s. convex. It can be approximated by a convex n-gon with the required accuracy for sufficiently large n.

Consider a pivotal polygonal section of a convex polyhedron.

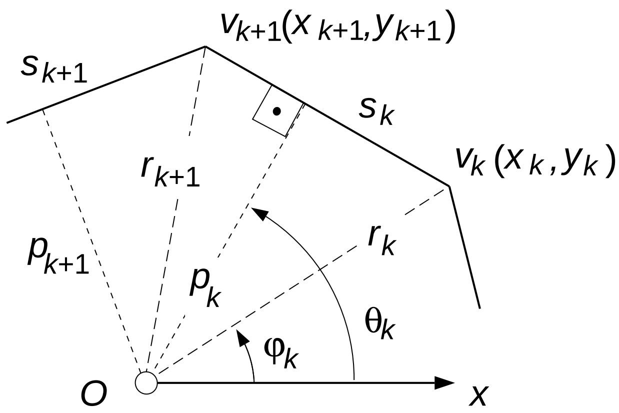


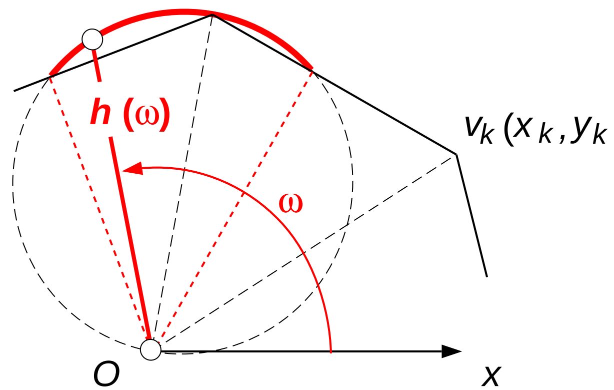
Convex polyhedron Pivotal section Flower

A useful representation of the flower of a convex polygon K :

$$H_K = \bigcup_{z \in \partial K} B_2(z/2, \|z\|/2), \; (O \in K^{\mathrm{o}}).$$

P Calka's formula for the flower area of a convex polygon





$$A(H_K) = \frac{1}{8}\sum_{k=1}^{n} R_{k+1}^2 \psi(Z_k, Z_{k+1}, F_{k+1}),$$

$\{(r_i, \varphi_i)\} :=$ polar coords. of the vertices, $\{(p_i, \theta_i)\} :=$ normal coords. of the sides,

$R := (\{r_i\}_1^n, \{r_i\}_1^n), \quad F := (\{\varphi_i\}_1^n, \{\varphi_i\}_1^n), \quad Z_i := (\{\theta_i\}_1^n, \{\theta_i + 2\pi\}_1^n),$

$\psi(\omega_1, \omega_2, \varphi) := \sin(2(\omega_2 - \varphi)) - \sin(2(\omega_1 - \varphi)) + 2(\omega_2 - \omega_1),$

$v_1(x_1, y_1)$ is the rightmost vertex.

(Calka (2003) *AAP* 35, 27–46, Calka (2009) personal communication).

P Calka's formula: Hint of proof

When K is a convex polygon its flower H_K is

$$H_K = \bigcup_{k=1}^{n} B_2((r_k/2, \varphi_k), r_k/2),$$

namely the union of the n disks whose diameters are the radius vectors of the vertices. Consequently,

$$h_K(\omega) = r_{k+1} \cos(\omega - \varphi_{k+1}),\ \omega \in [\theta_k, \theta_{k+1}),$$

and,

$$A(H_K) = \frac{1}{2} \sum_{k=1}^{n} \int_{Z_k}^{Z_{k+1}} h_K^2(\omega)\, \mathrm{d}\omega$$

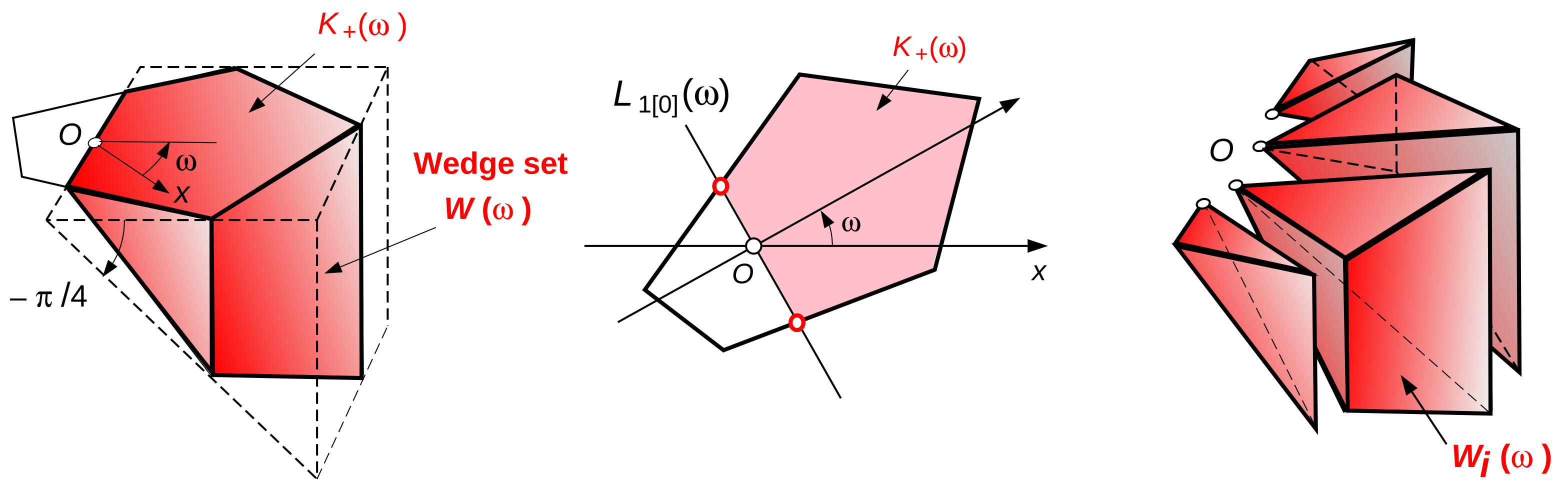
$$= \frac{1}{2} \sum_{k=1}^{n} \int_{Z_k}^{Z_{k+1}} R_{k+1}^2 \cos^2(\omega - F_{k+1})\, \mathrm{d}\omega,$$

from which the result follows

Mean wedge volume of a convex polygon (i)

$$V(Y) = 2\pi\,\mathbb{E}\{V(W_t)\}, \quad V(W_t) := \mathbb{E}V\{W(\omega;t)|t\}$$

Purpose: To find an exact formula for $V(W_t)$.
First step: For fixed $t \in \mathbb{S}^2_+$ and for a given $\omega \in [0, 2\pi)$, compute $V\{W(\omega;t)\}$ by a simplicial decomposition:



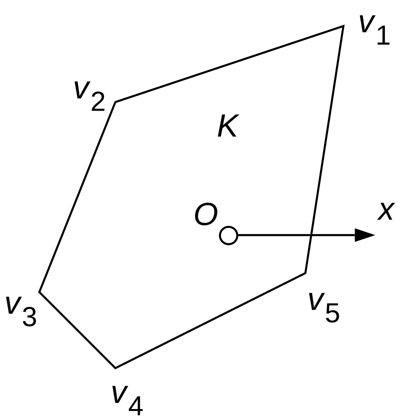
$W(\omega) := W(\omega;t)$

Mean wedge volume of a convex polygon (ii)

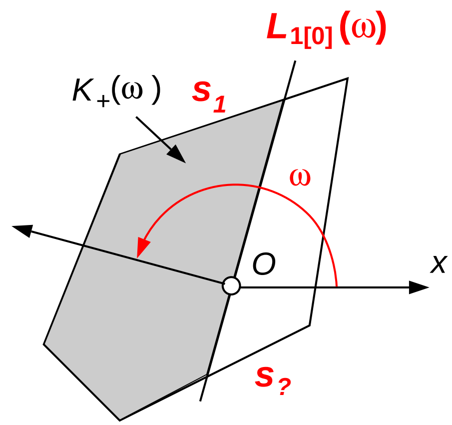
Second step: Compute

$$\mathbb{E}V\{W(\omega)\} = \frac{1}{2\pi}\int_0^{2\pi} V\{W(\omega)\}\,\mathrm{d}\omega.$$

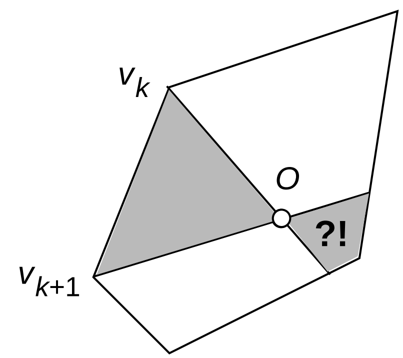
The main task is to identify the integration subranges. This is equivalent to identifying the vertices of the subpolygon $K_+(\omega)$ for each $\omega \in [0, 2\pi)$.



Convex *n*-gon



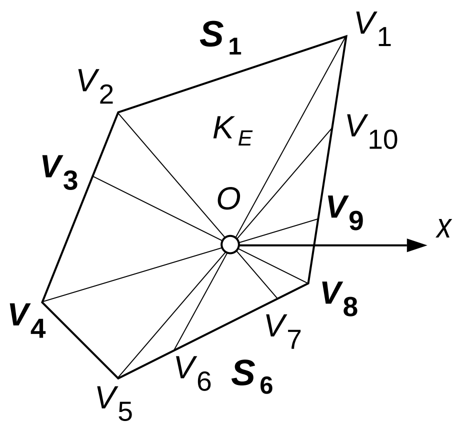
Subpolygon identification problem



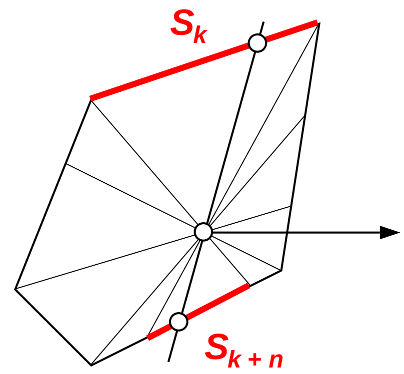
Subrange identification problem

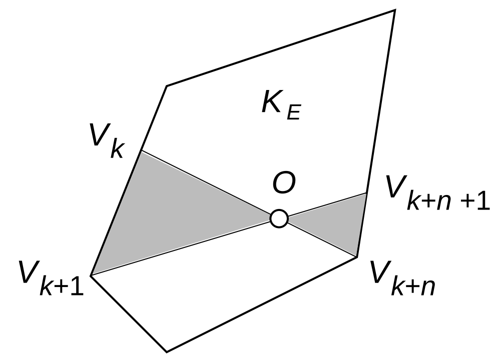
Mean wedge volume of a convex polygon (iii)

A tool to solve the identification problems: The 'extended polygon'.



'Extended polygon' K_E of K
(2n 'vertices')





No identification problems anymore

Mean wedge volume of a convex polygon (iv): Exact formula

The mean wedge volume associated with a convex $n-$gon K whose interior contains the origin can be expressed as follows,

$$2\pi\,\mathbb{E}_\omega V\{W(\omega)\} = \frac{1}{6}\sum_{k=1}^{2n}\Big\{P_k R_{k+1}^2 \psi_1(F_k, F_{k+1}, F_{k+1}, Z_k)$$

$$+\sum_{i=k+1}^{k+n-1} P_i L_i \cdot [R_i \psi_2(F_k, F_{k+1}, F_i) + R_{i+1}\psi_2(F_k, F_{k+1}, F_{i+1})]$$

$$- P_{k+n} R_{k+n}^2 \psi_1(F_k, F_{k+1}, F_{k+n}, Z_{k+n})\Big\},$$

where the angle vectors F, Z and the length vectors R, P, L are 4*n*-vectors defined on the extended polygon, whereas

$$\psi_1(\omega_1, \omega_2, \phi, \theta) := \sin(2\phi - \theta - \omega_2) - \sin(2\phi - \theta - \omega_1)$$

$$+ \cos^2(\phi - \theta)\cdot \log\left(\frac{\tan((\omega_2 - \theta)/2 + \pi/4)}{\tan((\omega_1 - \theta)/2 + \pi/4)}\right),$$

$$\psi_2(\omega_1, \omega_2, \phi) := \cos(\omega_2 - \phi) - \cos(\omega_1 - \phi). \qquad \square$$

Mean wedge volume of a convex polygon (iv): Approximate formula for large *n*

Monte Carlo integration with N steps:

$$2\pi\,\widehat{\mathbb{E}}_\omega V\{W(\omega)\} = \frac{2\pi}{N}\sum_{i=1}^{N} V\{W(\omega_i)\}, \qquad \omega_i = (U+i-1)\cdot\frac{2\pi}{N}+\frac{\pi}{2},\ i=1,2,...,N, \qquad U\sim \mathrm{UR}(0,1).$$

An unbiased estimator of the mean wedge volume associated with a convex $n-$gon K whose interior contains the origin can be computed as follows,

$$2\pi\,\widehat{\mathbb{E}}_\omega V\{W(\omega)\} = \frac{2\pi}{N}\cdot\frac{1}{6}\sum_{i=1}^{N}\Big\{P_{k(i)}R^2_{k(i)+1}\psi\Big(\omega_i, F_{k(i)+1}, Z_{k(i)}\Big)$$

$$+\sum_{j=k(i)+1}^{k(i)+n-1} P_jL_j\cdot\left[R_j\cos\left(\omega_i - F_j\right) + R_{j+1}\cos\left(\omega_i - F_{j+1}\right)\right]$$

$$-P_{k(i)+n}R^2_{k(i)+n}\psi\Big(\omega_i, F_{k(i)+n}, Z_{k(i)+n}\Big)\Big\},$$

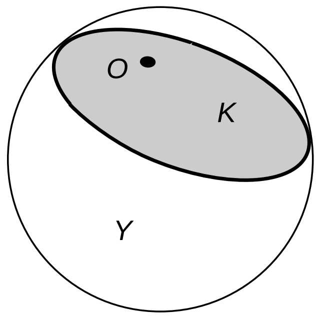
where the angle vectors F, Z and the length vectors R, P, L are defined as before; the index $k(i)$ is such that the axis $L_{1[0]}(\omega_i)$ hits the sides $S_{k(i)}$ and $S_{k(i)+n}$ of the extended polygon, and

$$\psi(\omega,\phi,\theta) := \frac{\cos^2(\omega-\phi)}{\sin(\omega-\theta)}.$$

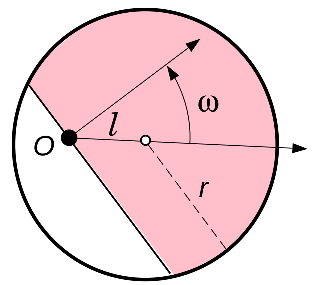
Furthermore,

$$\mathrm{Var}\Big\{\widehat{\mathbb{E}}_\omega V\{W(\omega)\}\Big\} = O\big(N^{-\alpha}\big), \quad \alpha\in[2,4].$$

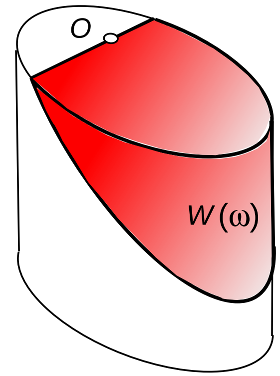
The mean wedge volume of a disk



Ball and pivotal section



Pivotal section



Wedge

An UE of the volume of the ball is,

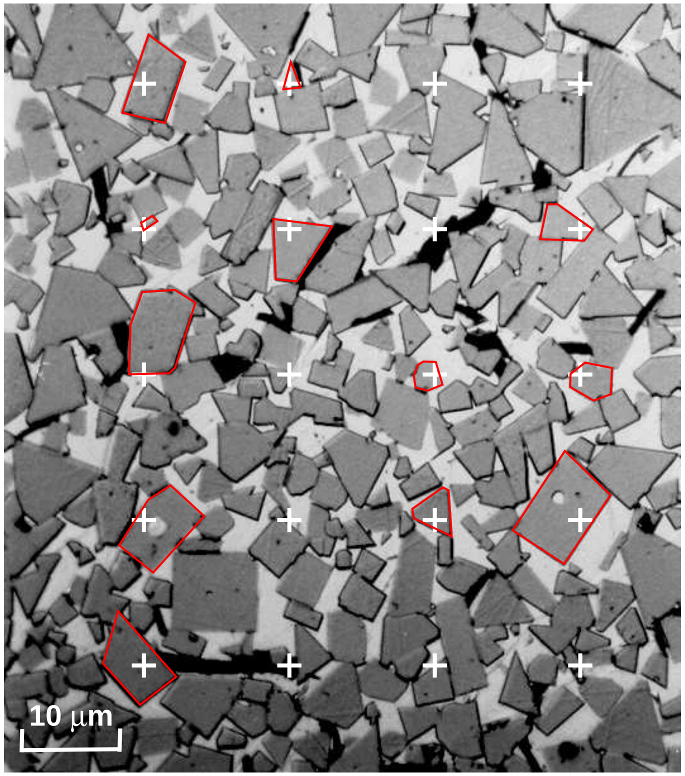
$$\widehat{V}(Y;\lambda) = 2\pi\mathbb{E}_\omega V\{W(\omega)\} = \frac{8}{9}r^3\left[\left(7+\lambda^2\right)E\left(\lambda^2\right) - 4\left(1-\lambda^2\right)K\left(\lambda^2\right)\right],\ \ \lambda = l/r \in (0,1),$$

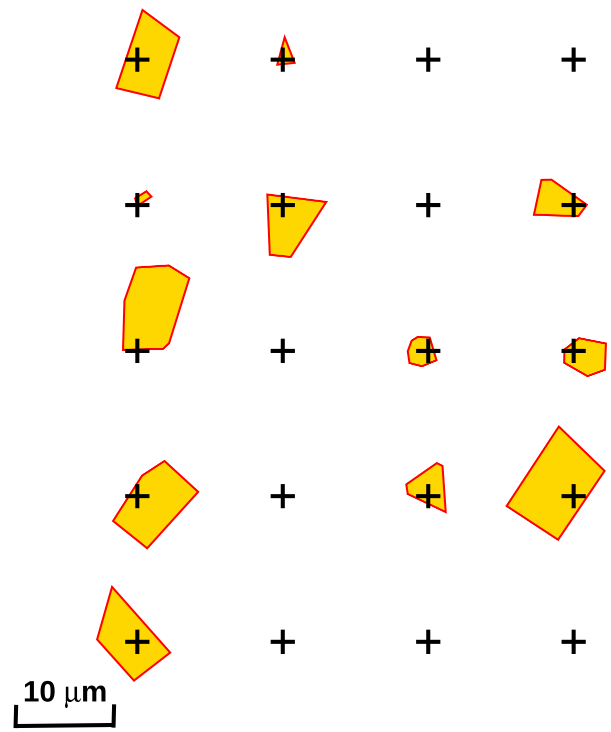
where K, E represent the complete elliptic integrals of the first and second kinds, respectively. Note that $\widehat{V}(Y;0) = (4/3)\pi r^3$.

Approximating the disk by a regular polygon of $n = 5000$ vertices, the Monte Carlo approximation yielded 6 exact digits of $\widehat{V}(Y;\lambda)$ for $\lambda = 0.1, 0.3, 0.7, 0.9$, with $N = 20$ steps. Real computing time: 2s. With the exact formula: 102s.

Mean volume weighted surface area and volume of cemented carbide grains (i)

Material. Cemented carbides constitute a group of powder metallurgical materials produced by sintering, and characterized by a very high hardness and wear resistance. Their microstructure consists of hard tungsten carbide (WC) grains embedded in a ductile cobalt (Co) matrix.





$$\mathbb{E}_W(X) = \frac{\mathbb{E}_N(WX)}{\mathbb{E}_N(W)},\ \text{here we want } \mathbb{E}_V(S),\ \mathbb{E}_V(V).$$

Mean volume weighted surface area and volume of cemented carbide grains (ii)

Let $1 \leq p < \infty$ be the number of test points hitting grains, and let s_i, v_i denote unbiased estimators of the surface area and the volume of the grain hit by the $i-$th test point, respectively, $(i = 1, 2, ..., p)$. Because the mentioned grain is volume-weighted, the estimators:

$$\overline{s}_V = \frac{1}{p}\sum_{i=1}^{p} s_i = \frac{4}{p}\sum_{i=1}^{p} F_i, \text{ where } F_i := \text{ flower area,}$$

$$\overline{v}_V = \frac{1}{p}\sum_{i=1}^{p} v_i = \frac{2\pi}{p}\sum_{i=1}^{p} W_i, \text{ where } W_i := \text{ mean wedge volume,}$$

are ratio unbiased for the target volume weighted means $\mathbb{E}_V S$ and $\mathbb{E}_V V$, respectively. In shorthand,

$$\overline{s}_V = 4\overline{F},$$

$$\overline{v}_V = 2\pi\overline{W}.$$

Mean volume weighted surface area and volume of cemented carbide grains (iii)

Results for the cemented carbide grains.

The 1997 results were reported in Karlsson & C–O (1997). For the "surfactor" and "PSI" (point sampled intercepts) methods, see the mentioned study.

Quadrat	# polygons	$\overline{A}$, $\mu\mathrm{m}^2$	$\bar{s}_V = 4\overline{F}$, $\mu\mathrm{m}^2$	$\bar{v}_V = 2\pi\overline{W}$, $\mu\mathrm{m}^3$
1	20	21.24	144.9	135.5
2	17	32.36	209.3	233.6
3	16	24.76	160.1	160.3
4	17	21.44	142.5	124.9
Pool	70	24.80 (10.4%)	163.4 (9.5%)	162.4 (15.0%)
Results of 1997			$\bar{s}_V$ $\mu\mathrm{m}^2$, surfactor	$\bar{v}_V$ $\mu\mathrm{m}^3$, PSI
			154.5 (4.4%)	161.3(23.7%)

Uniqueness properties of the invariator

The mean wedge volume is the averaged nucleator

Bounded particle: $Y \subset B_3 \subset \mathbb{R}^3$.

Mean wedge volume representation of the particle volume:

$$V(Y) = 2\pi\, \mathbb{E}_t \mathbb{E}_\omega V\{W(\omega; t)\}.$$

Nucleator representation (to simplify the notation suppose for the moment that Y is star shaped with respect to an interior pivotal point O):

$$V(Y) = \frac{4\pi}{3} \mathbb{E}_t \mathbb{E}_\varphi \Big\{ \rho_+^3(\varphi; t) \Big\}.$$

Proposition 1. *For any pivotal section of a particle with respect to an interior pivotal point, namely for each orientation $t \in \mathbb{S}_+^2$ of the pivotal plane, the following identity holds,*

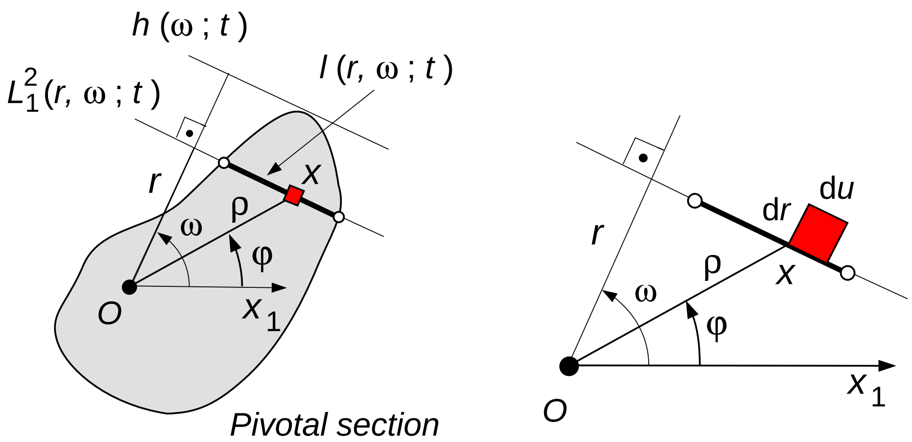
$$\mathbb{E}_\omega V\{W(\omega; t)\} = \frac{2}{3}\, \mathbb{E}_\varphi \Big\{ \rho_+^3(\varphi; t) \Big\},$$

that is, the mean wedge volume coincides with 2/3 times the averaged third power of the nucleator ray length.

Proof. For each $t \in \mathbb{S}^2_+$ and each $\omega \in [0, 2\pi)$,

$$V\{W(\omega;t)\} = \int_0^{h(\omega;t)} l(r,\omega;t)\, r \mathrm{d}r.$$

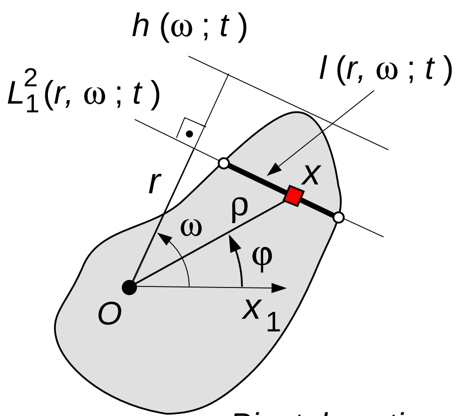
Consider a point of unsigned abscissa u along the $p-$line with respect to an arbitrary origin on this line. Then clearly,



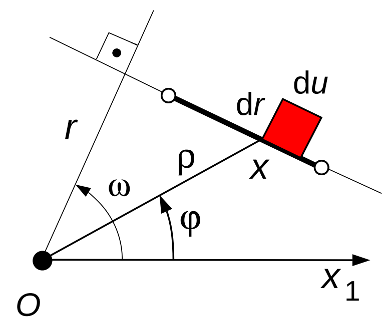
$$l(r,\omega;t) = \int_{\left(Y \cap L_2^3(0,t)\right) \cap L_1^2(r,\omega;t)} \mathrm{d}u,$$

and the length elements $\mathrm{d}u$ and $\mathrm{d}r$ are orthogonal. Thus $\mathrm{d}x := \mathrm{d}u\, \mathrm{d}r$ is the area element at a point x in the pivotal plane. Let (ρ, φ), $(\rho > 0, 0 \le \varphi < 2\pi)$ denote the polar coordinates of x with respect to the fixed axis Ox_1.

Then $r = \rho|\cos(\omega - \varphi)|$ and we can write,



Pivotal section



$$\mathbb{E}_\omega V\{W(\omega;t)\} = \frac{1}{2\pi}\int_0^\pi \mathrm{d}\omega \int_{Y\cap L_2^3(0,t)} \rho|\cos(\omega-\varphi)|\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\int_{Y\cap L_2^3(0,t)} \rho\,\mathrm{d}x \int_0^\pi |\cos(\omega-\varphi)|\mathrm{d}\omega$$

$$= \frac{1}{\pi}\int_{Y\cap L_2^3(0,t)} \rho\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\rho_+(\varphi;t)} \rho^2\,\mathrm{d}\rho$$

$$= \frac{2}{3}\,\mathbb{E}_\varphi\left\{\rho_+^3(\varphi;t)\right\}.$$

Computation of the mean wedge volume defined on a convex polygon via the averaged nucleator: a vast improvement on Cruz-Orive (2011)

Corollary 1. *For a convex $n-$gon $K \subset \mathbb{R}^2$ the mean volume of the wedge set $W_K(\omega)$ defined on K with respect to an arbitrary pivotal point in its plane may be expressed as follows,*

$$2\pi\, \mathbb{E}_\omega V\{W_K(\omega)\} = \frac{1}{3}\sum_{k=1}^{n} p_k^3 \cdot \psi(\varphi_k, \varphi_{k+1}, \theta_k),$$

where

$$\psi(\omega_1, \omega_2, \theta) := \frac{\sin(\omega_2-\theta)}{\cos^2(\omega_2-\theta)} - \frac{\sin(\omega_1-\theta)}{\cos^2(\omega_1-\theta)} + \log\left(\frac{\tan((\omega_2-\theta)/2+\pi/4)}{\tan((\omega_1-\theta)/2+\pi/4)}\right).$$

If $O \notin K^{\mathrm{o}}$, then K must be contained in the upper half plane.

Proof. For $k = 1, 2, ..., n$ let $\rho_k(\omega)$, $\omega \in [\varphi_k, \varphi_{k+1})$ denote the radial function of ∂K. Bearing in mind that $\rho_k(\omega)\cos(\omega-\theta_k) = p_k$, by virtue of Proposition 1 we have,

$$2\pi\, \mathbb{E}_\omega V\{W(\omega;t)\} = \frac{2}{3}\sum_{k=1}^{n}\int_{\varphi_k}^{\varphi_{k+1}} \rho_k^3(\omega)\,\mathrm{d}\omega = \frac{2}{3}\sum_{k=1}^{n} p_k^3 \int_{\varphi_k}^{\varphi_{k+1}} \frac{\mathrm{d}\omega}{\cos^3(\omega-\theta_k)}.$$

On the other hand,

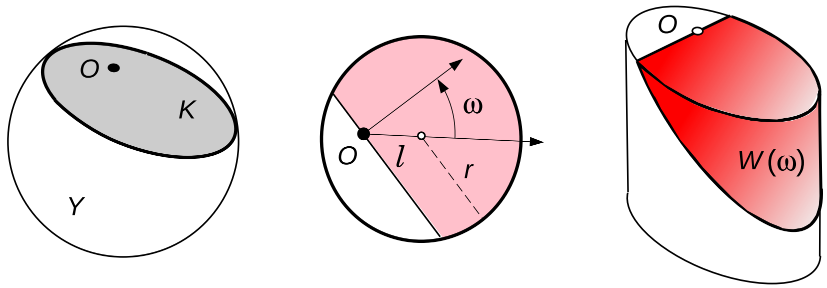
$$\int \frac{\mathrm{d}x}{\cos^3 x} = \frac{1}{2}\frac{\sin x}{\cos^2 x} + \frac{1}{2}\log\tan\left(\frac{x}{2}+\frac{\pi}{4}\right) + C,$$

whereby Corollary 1 follows.

Checking the new mean wedge volume formula

Assume that the particle $Y \subset \mathbb{R}^3$ is a finite ball. The following exact results have been checked numerically with the formula.

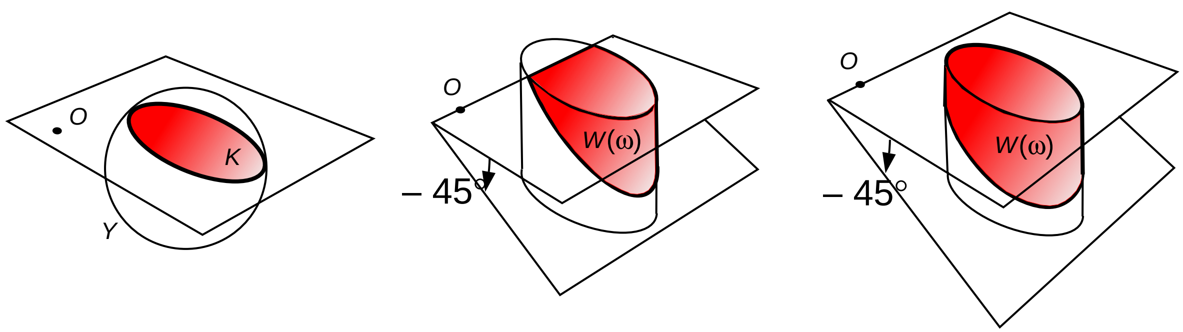
Case of an interior pivotal point



Ball and pivotal section   Pivotal section   Wedge

$$2\pi\mathbb{E}_\omega V\{W(\omega)\} = \frac{8}{9}r^3\left[\left(7+\lambda^2\right)E\left(\lambda^2\right) - 4\left(1-\lambda^2\right)K\left(\lambda^2\right)\right], \quad \lambda := l/r \in (0,1)$$

Case of an exterior pivotal point



Ball and pivotal section   Wedges with exterior pivotal point *O*

$$2\pi\mathbb{E}_\omega V\{W(\omega)\} = \frac{8}{9\lambda}\left[\lambda^2\left(7+\lambda^2\right)E\left(\lambda^{-2}\right) - \left(\lambda^2+3\right)\left(\lambda^2-1\right)K\left(\lambda^{-2}\right)\right], \quad \lambda := l/r > 1$$

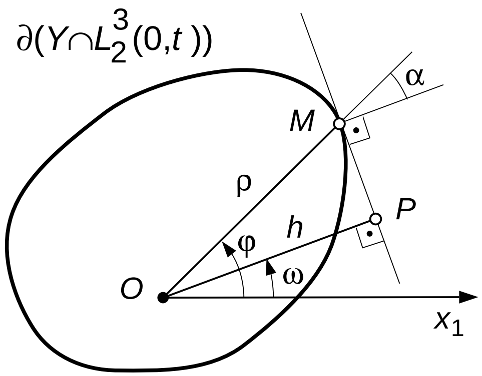
The integrated surfactor is the flower area

For a convex particle $Y \subset \mathbb{R}^3$ containing the pivotal point, the flower formula for the particle surface area reads,

$$S(\partial Y) = 4\,\mathbb{E}_t\{A(H_t)\} = 2\,\mathbb{E}_t\left\{\int_0^{2\pi} h^2\,\mathrm{d}\omega\right\}$$

Alternatively, the surfactor representation is,

$$S(\partial Y) = 4\pi\,\mathbb{E}_t\mathbb{E}_\varphi\left\{\rho^2(1+\alpha\,\tan\alpha)\right\} = 2\,\mathbb{E}_t\left\{\int_0^{2\pi}\rho^2(1+\alpha\,\tan\alpha)\,\mathrm{d}\varphi\right\}$$



The next proposition states that the two expressions in curly brackets coincide for each $t \in \mathbb{S}^2$.

Case of a convex particle with smooth boundary

Proposition 2. *For each pivotal section of a convex particle with boundary of class C^2 containing the pivotal point, the following identity holds,*

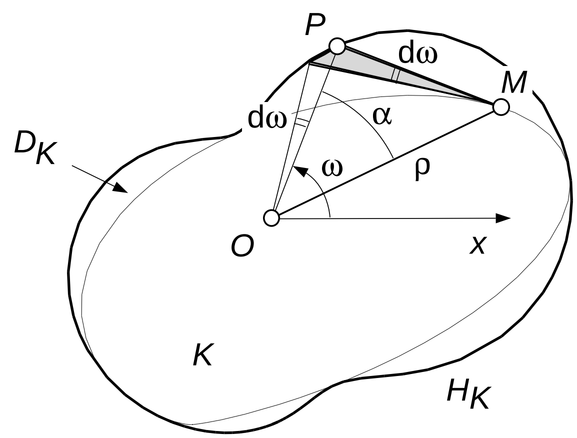
$$\int_0^{2\pi} \rho^2(1+\alpha\,\tan\alpha)\,\mathrm{d}\varphi = \int_0^{2\pi} h^2\,\mathrm{d}\omega.$$

Lemma 1. *The flower area $A(H_K)$ of a planar convex set K with boundary of class C^2 containing the origin may be expressed as follows,*

$$A(H_K) = A(K) + \frac{1}{2}\int_0^{2\pi} \rho^2\,\sin^2\alpha\,\mathrm{d}\omega,$$

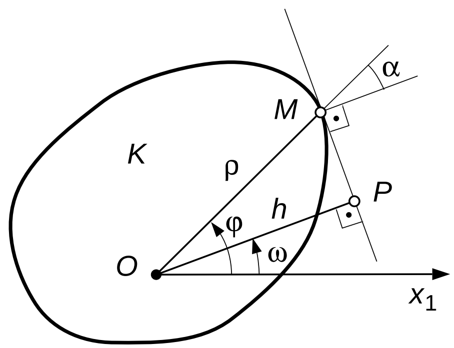
where the angle α is defined as in Eq. 24. Equivalently,

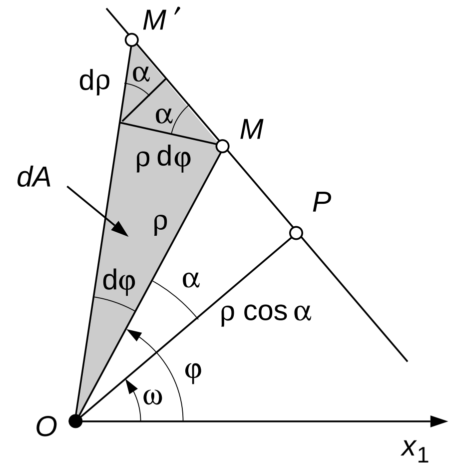
$$\int_0^{2\pi} h^2\,\mathrm{d}\omega = \int_0^{2\pi} \rho^2\,\mathrm{d}\varphi + \int_0^{2\pi} \rho^2\,\sin^2\alpha\,\mathrm{d}\omega.$$



Proof 1 of Lemma 1. Clear from the figure.

Proof 2 of Lemma 1. The ordinary area element $\mathrm{d}A$ in polar coordinates is $(1/2)\,\rho^2\,\mathrm{d}\varphi$. However, for the present purposes we need a different representation. The area element may be regarded as an elementary triangle OMM' of height $\rho\cos\alpha$ and base $\sin\alpha\,\mathrm{d}\rho + \rho\cos\alpha\,\mathrm{d}\varphi$, see the figure.





Therefore,

$$\mathrm{d}A = (1/2)\rho\sin\alpha\cos\alpha\,\mathrm{d}\rho + (1/2)\rho^2\cos^2\alpha\,\mathrm{d}\varphi = (1/2)\,\rho\rho'_\alpha\sin\alpha\cos\alpha\,\mathrm{d}\alpha + (1/2)\,\rho^2\cos^2\alpha\,\mathrm{d}\varphi,$$

where $\rho'_\alpha := \mathrm{d}\rho/\mathrm{d}\alpha$. Thus,

$$2A(K) = \int_0^{2\pi}\rho\rho'_\alpha\sin\alpha\cos\alpha\,\mathrm{d}\alpha + \int_0^{2\pi}\rho^2\cos^2\alpha\,\mathrm{d}\varphi.$$

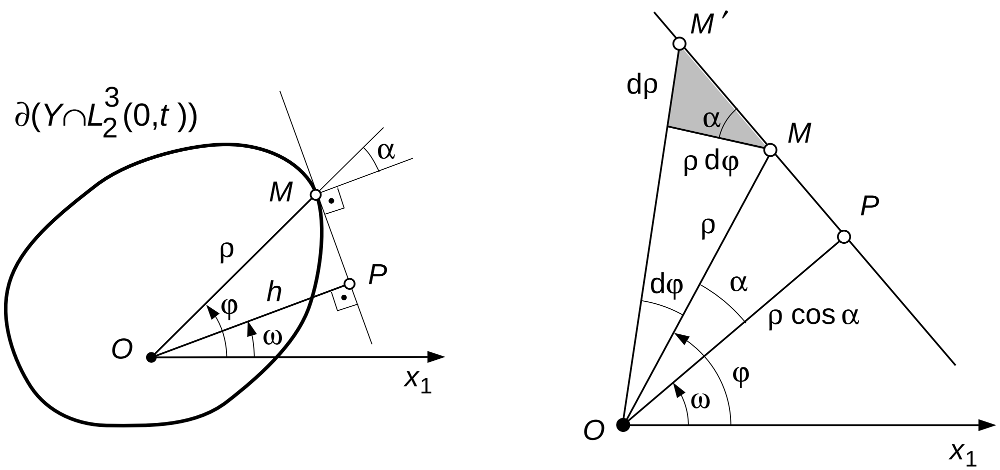
Integration by parts with $u = \sin\alpha\cos\alpha$ and $\mathrm{d}v = \rho\rho'_\alpha\,\mathrm{d}\alpha$ yields,

$$\int_0^{2\pi}\rho\rho'_\alpha\sin\alpha\cos\alpha\,\mathrm{d}\alpha = -\frac{1}{2}\int_0^{2\pi}\rho^2\Big(\cos^2\alpha - \sin^2\alpha\Big)\,\mathrm{d}\alpha.$$

Recalling that $\alpha = \varphi - \omega$ and $\rho\cos\alpha = h$, simplification leads to the required result.

Proof of Proposition 2. From the right hand side figure below we obtain the key relation

$$\tan\alpha = \frac{\mathrm{d}\rho}{\rho\,\mathrm{d}\varphi} = \frac{\rho'_\varphi}{\rho}.$$



Therefore,

$$\int_0^{2\pi} \rho^2\alpha\,\tan\alpha\,\mathrm{d}\varphi = \int_0^{2\pi} \rho\rho'_\varphi\alpha\;\mathrm{d}\varphi.$$

Integration by parts with $u = \alpha$ and $\mathrm{d}v = \rho\rho'_\varphi\,\mathrm{d}\varphi$ yields,

$$\int_0^{2\pi} \rho^2\alpha\,\tan\alpha\,\mathrm{d}\varphi = -\frac{1}{2}\int_0^{2\pi} \rho^2\;\mathrm{d}\alpha = -\frac{1}{2}\int_0^{2\pi} \rho^2\;\mathrm{d}\varphi + \frac{1}{2}\int_0^{2\pi} \rho^2\Big(\sin^2\alpha + \cos^2\alpha\Big)\,\mathrm{d}\omega.$$

Bearing in mind that $\rho \cos\alpha = h$

$$\int_0^{2\pi} \rho^2 (1 + \alpha \tan\alpha)\, \mathrm{d}\varphi = \int_0^{2\pi} \rho^2\, \mathrm{d}\varphi - \frac{1}{2}\int_0^{2\pi} \rho^2\; \mathrm{d}\varphi + \frac{1}{2}\int_0^{2\pi} \rho^2 \sin^2\alpha\; \mathrm{d}\omega + \frac{1}{2}\int_0^{2\pi} h^2\, \mathrm{d}\omega$$

$$= \frac{1}{2}\int_0^{2\pi} h^2\, \mathrm{d}\omega - \frac{1}{2}\int_0^{2\pi} \rho^2 \sin^2\alpha\; \mathrm{d}\omega + \frac{1}{2}\int_0^{2\pi} \rho^2 \sin^2\alpha\; \mathrm{d}\omega + \frac{1}{2}\int_0^{2\pi} h^2\, \mathrm{d}\omega \quad \text{(Lemma 1)}$$

$$= \int_0^{2\pi} h^2\, \mathrm{d}\omega. \qquad \square$$

CASE OF A CONVEX POLYHEDRAL PARTICLE

Proposition 2 has been proved under the assumption that the particle Y has a unique tangent plane at every point of its boundary. Because this property may look restrictive, here we assume that Y is a convex polyhedron. Thus, every pivotal section is almost surely a convex polygon.

Proposition 3. *Proposition 2 holds also for any convex n–gon with an interior pivotal point.*

Proof. Recalling that $\rho_k(\omega)\cos(\omega - \theta_k) = p_k$, we can write,

$$\int_0^{2\pi} \rho^2(1 + \alpha\ \tan\alpha)\,\mathrm{d}\varphi = \sum_{k=1}^{n} \int_{\varphi_k}^{\varphi_{k+1}} \frac{p_k^2}{\cos^2\alpha}(1 + \alpha\tan\alpha)\,\mathrm{d}\varphi,$$

where $\alpha := \varphi - \theta_k$ in the preceding integrand. On the other hand, by Calka's formula,

$$\int_0^{2\pi} h^2\,\mathrm{d}\omega = \frac{1}{4}\sum_{k=1}^{n} r_{k+1}^2[\zeta(\theta_{k+1} - \varphi_{k+1}) - \zeta(\theta_k - \varphi_{k+1})],$$

$$\zeta(x) := 2x + \sin(2x).$$

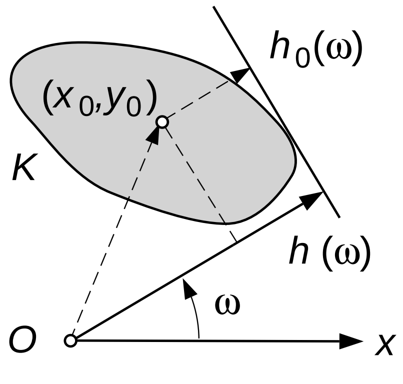
Thus, the right hand sides of the preceding two equations must coincide. This is readily verified on substituting the following results

$$\int \frac{1 + x\tan x}{\cos^2 x}\,\mathrm{d}x = \frac{1}{2}\tan x + \frac{1}{2}\frac{x}{\cos^2 x} + C,$$

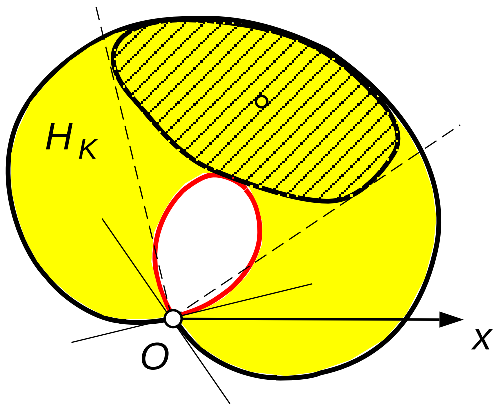
$$p_k = r_k\cos(\theta_k - \varphi_k) = r_{k+1}\cos(\theta_k - \varphi_{k+1}),$$

into the right hand side of the former equation, and simplifying.

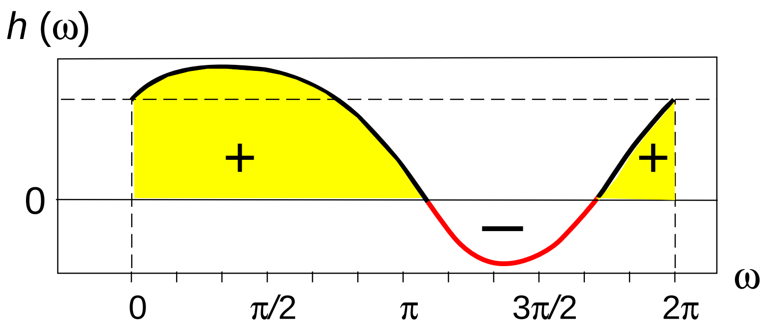
Flower of a convex set with exterior pivotal point: A glimpse



Support function $h\ (\omega)$ with exterior pivotal point O



Support set (flower) H_K

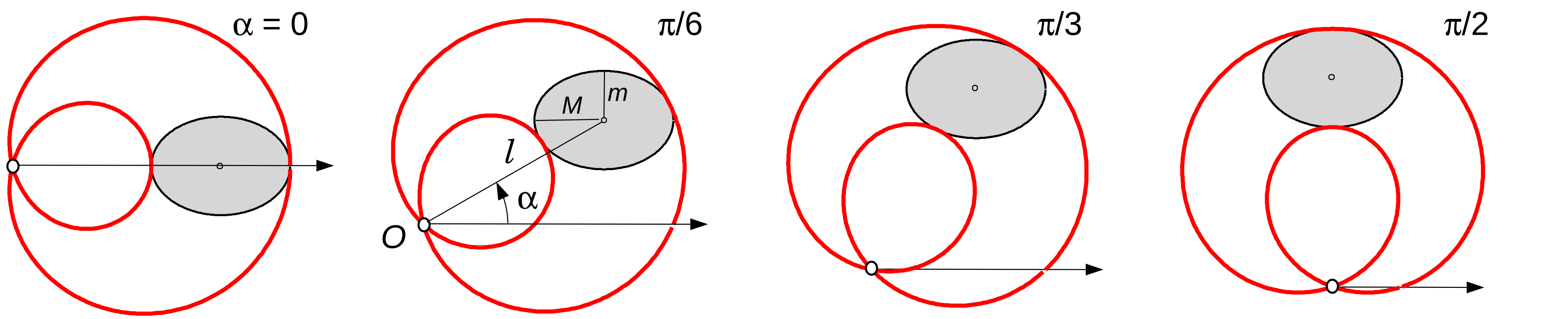


Unwrapped graph of $h\ (\omega)$

$$h(\omega) = x_0 \cos\omega + y_0 \sin\omega + h_0(\omega), \quad \omega \in [0, 2\pi),$$

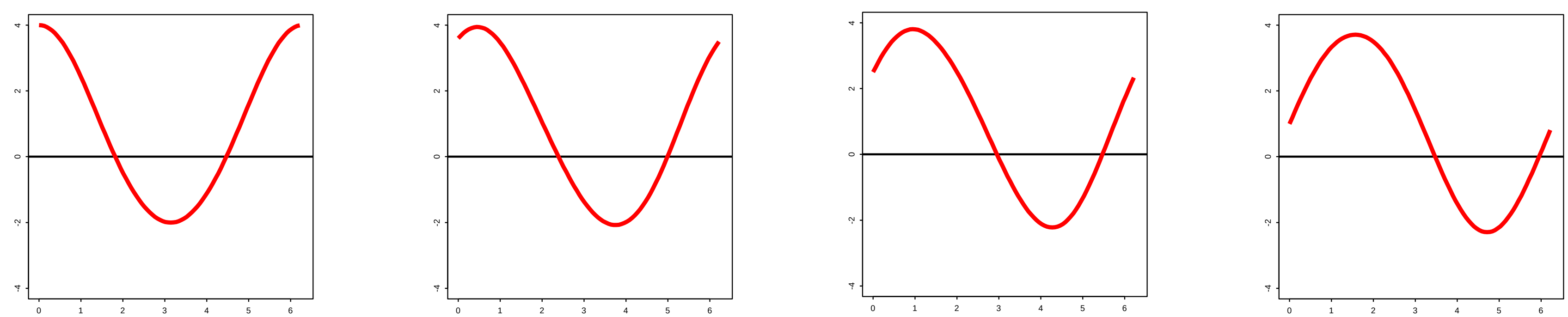
$$A(H_K) = \frac{1}{2}\int_0^{2\pi} h(\omega)\,|h(\omega)|\,\mathrm{d}\omega.$$

Flowers of an ellipse with exterior pivotal point



$l = 3$, $M = 1$, $m = 1/\sqrt{2}$

$A(H_K) =$ 10.9869 | 10.7190 | 10.2063 | 9.9597



With an exterior pivotal point, the flower area of an ellipse depends not only on M, m, l, but also on α.

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