Recent results on stereology

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Part I

1. Classical construction of motion invariant test probes. Crofton formulae. Applications to stereology. Exercise 1.

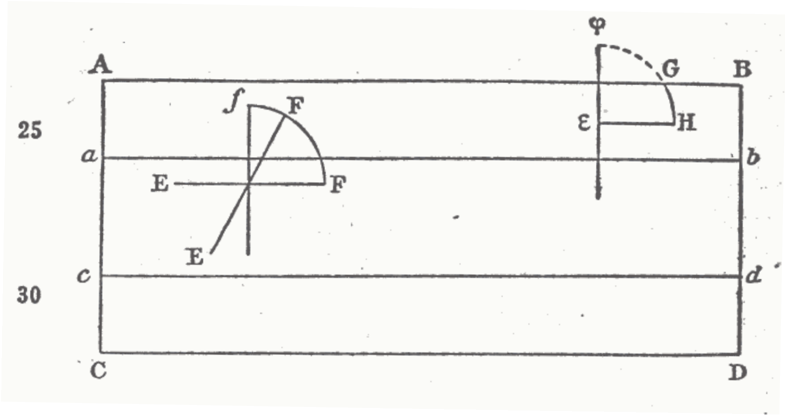
2. Motion invariant test lines in $\mathbb{R}^3$. Invariator construction. Applications to stereology. Exercise 2.

3. Case of a convex particle. Surface area in terms of the flower set.

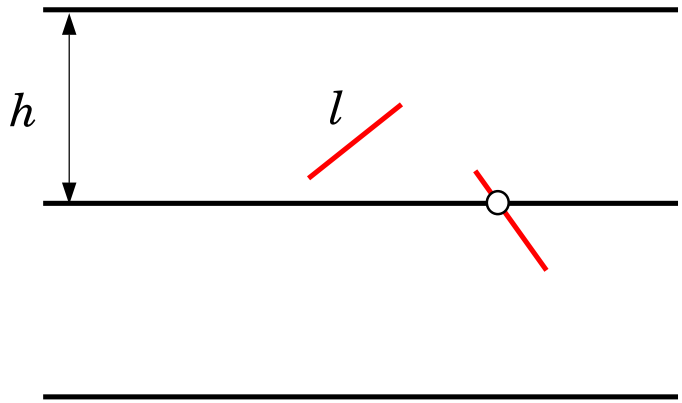
Classical construction of motion invariant test probes. Crofton formulae. Applications to stereology.

Invariant test probes in $\mathbb{R}^2$: Buffon's needle problem

George Louis Leclerc,
Comte de Buffon (1707-1788)



Buffon's needle problem (1777)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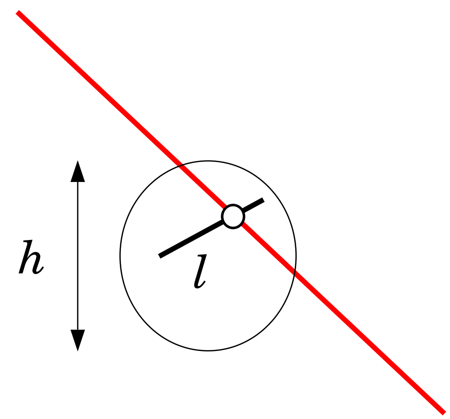
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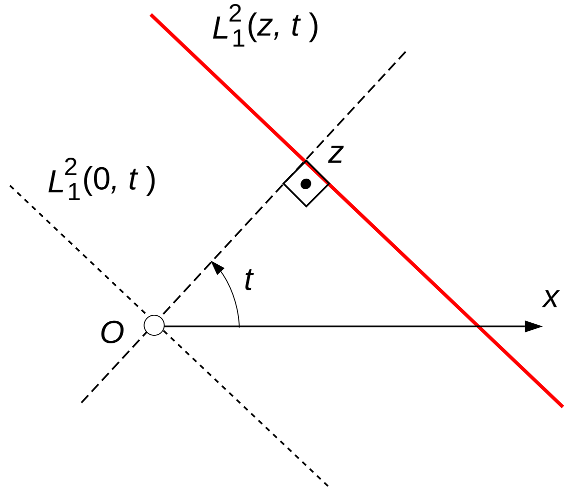
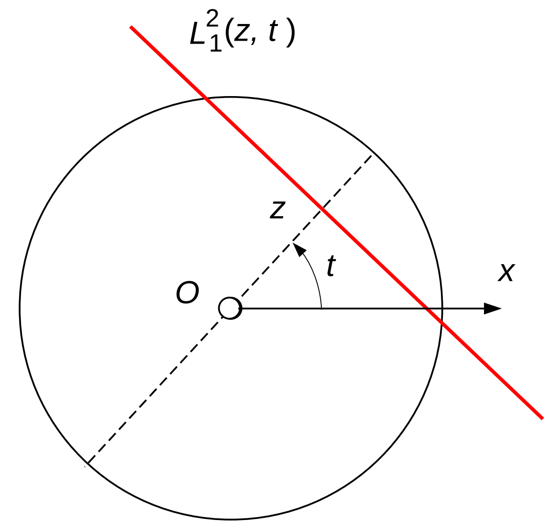




$L_1^2(z, t) = L_1^2(0, t) + z$

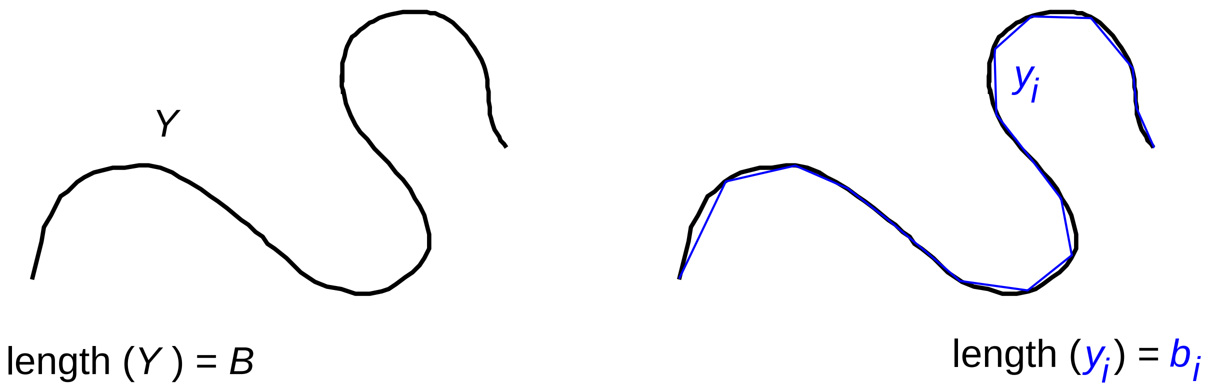
Invariant density for straight lines in $\mathbb{R}^2$:

$$\mathrm{d}L_1^2 = \mathrm{d}z \wedge \mathrm{d}t, \ z \in \mathbb{R}, \ t \in \mathbb{S}_+^1 := [0, \pi).$$

For straight lines hitting a disk of diameter h, the joint probability element of (z, t) is,

$$\mathbb{P}(\mathrm{d}z, \mathrm{d}t) = \frac{\mathrm{d}z}{h} \cdot \frac{\mathrm{d}t}{\pi}, \ z \in [-h/2, h/2], \ t \in [0, \pi)$$

Buffon-Steinhaus estimation of curve length in $\mathbb{R}^2$. Preliminaries

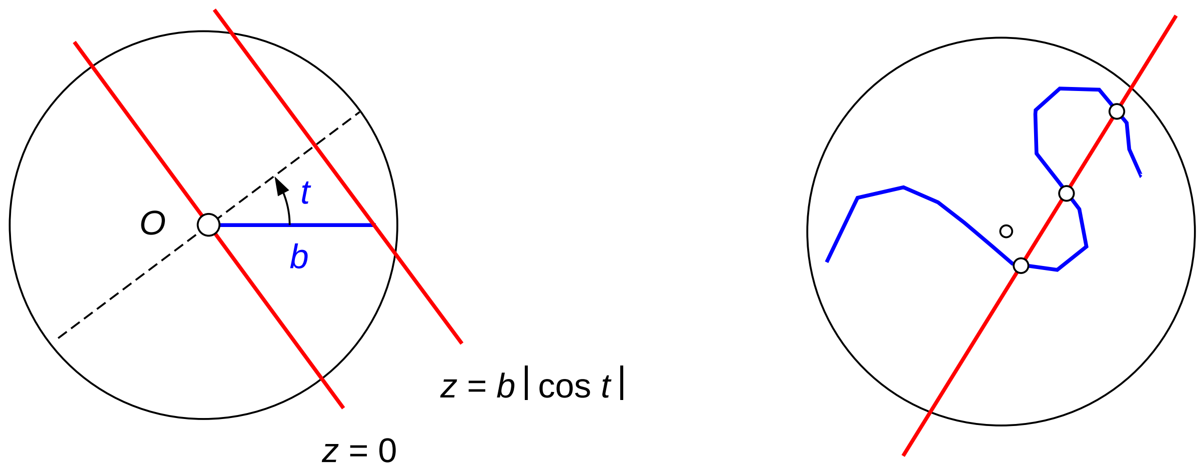


- Rectifiable curve: $Y \in \mathbb{R}^2$
- Target: $B := \mathrm{length}(Y)$

$$Y_n := \bigcup_{i=1}^{n} y_i, \quad y_i := \text{ straight line segment, length}(y_i) = b_i$$

$$B_n := \text{ length}(Y_n) = \sum_{i=1}^{n} b_i$$

$$B = \sup_{n \in \mathbb{N}} B_n$$



$$\mathbb{E}I\Big(y \cap L_1^2\Big) = \mathbb{P}\Big(L_1^2 \uparrow y\Big) = \int_{L_1^2 \uparrow y} \mathbb{P}(\mathrm{d}z, \mathrm{d}t) = \frac{1}{\pi h}\int_0^{\pi} \mathrm{d}t \int_0^{b|\cos t|} \mathrm{d}z = \frac{b}{\pi h}\int_0^{\pi} |\cos t|\,\mathrm{d}t = \frac{2b}{\pi h},$$ Buffon's answer,

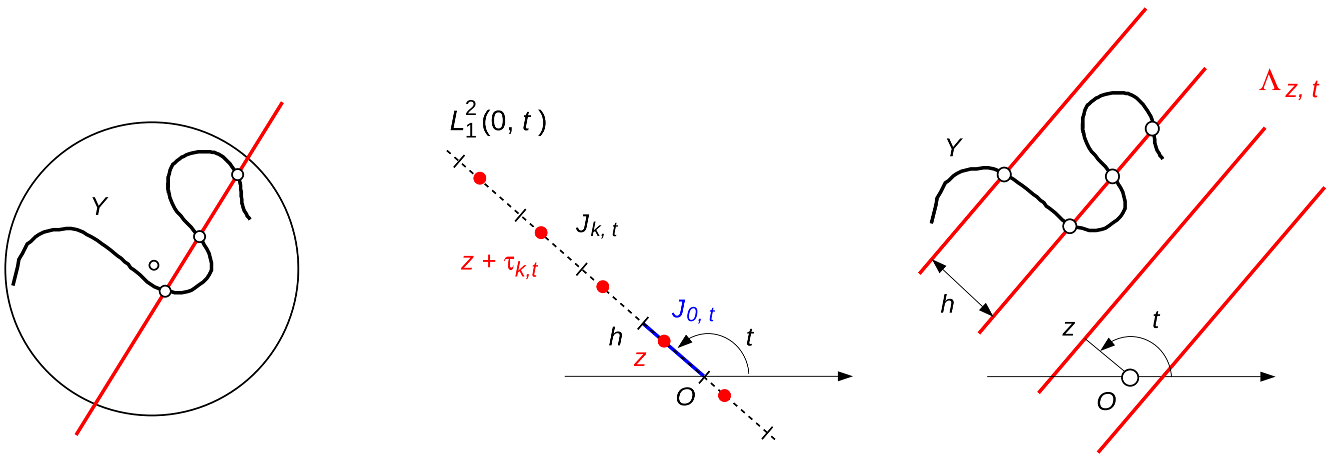
$$\mathbb{E}I\Big(Y_n \cap L_1^2\Big) = \mathbb{E}I\Big((\cup_1^n y_i) \cap L_1^2\Big) = \mathbb{E}\sum_{i=1}^n I\Big(y_i \cap L_1^2\Big) = \sum_{i=1}^n \mathbb{E}I\Big(y_i \cap L_1^2\Big) = \frac{2}{\pi h}\sum_{i=1}^n b_i = \frac{2B_n}{\pi h}$$

Apply the *Monotone Convergence Theorem (Beppo Levi)* (e.g. Bogachev, V.I. (2007) *Measure Theory*, Vol.1):

$$\mathbb{E}I\Big(Y \cap L_1^2\Big) = \mathbb{E}\Big\{\sup_{n\in\mathbb{N}} I\Big(Y_n \cap L_1^2\Big)\Big\} = \sup_{n\in\mathbb{N}} \mathbb{E}I\Big(Y_n \cap L_1^2\Big) = \frac{2}{\pi h}\cdot \sup_{n\in\mathbb{N}} B_n = \frac{2B}{\pi h}$$

(Voss, F. (2005) *Diplomarbeit*, Univ. D-Ulm).

Buffon-Steinhaus test system



- Choose an isotropic orientation $t \in [0, \pi)$ using $\mathbb{P}(\mathrm{d}t) = \mathrm{d}t/\pi$, e.g. $t = U\pi, \quad U \sim \mathrm{UR}[0,1)$.

- On $L_1^2(0,t)$, fix a fundamental half open segment ('tile') $J_{0,t}$ of length h. Construct the partition $L_1^2(0,t) = \cup_{k\in\mathbb{Z}} J_{k,t}$, where $J_{k,t} = J_{0,t} + \tau_{k,t}$ and $-\tau_{k,t}$ is a translation of modulus h along $L_1^2(0,t)$ which brings the tile $J_{k,t}$ to coincide with $J_{0,t}$ leaving the partition invariant for each $k \in \mathbb{Z}$ and $t \in [0,\pi)$.

- Independently of t choose a uniform random point $z \in J_{0,t}$, namely with $\mathbb{P}(\mathrm{d}z) = \mathrm{d}z/h$, e.g. $t = Uh, \quad U \sim \mathrm{UR}[0,1)$.

- The system of parallel straight lines

$$\Lambda_{z,t} = \left\{ L_1^2(z + \tau_{k,t}, t),\ k \in \mathbb{Z} \right\},$$

  is a test system of parallel straight lines with motion invariant density.

Buffon-Steinhaus estimation of curve length in $\mathbb{R}^2$

Next, apply a particular version of *Santaló's theorem for test systems* (Santaló, L.A. (1976) *Integral Geometry and Geometric Probability*, Ch. 8),

$$2B = \int_0^\pi \mathrm{d}t \int_{\mathbb{R}} I\Big(Y \cap L_1^2(z,t)\Big)\,\mathrm{d}z = \int_0^\pi \mathrm{d}t \sum_{k\in\mathbb{Z}} \int_{J_{k,t}} I\Big(Y \cap L_1^2(z,t)\Big)\,\mathrm{d}z$$

$$= \int_0^\pi \mathrm{d}t \sum_{k\in\mathbb{Z}} \int_{J_{0,t}} I\Big(Y \cap L_1^2\big(z+\tau_{k,t},t\big)\Big)\,\mathrm{d}z$$

$$= \int_0^\pi \mathrm{d}t \int_{J_{0,t}} I(Y \cap \Lambda_{z,t})\,\mathrm{d}z$$

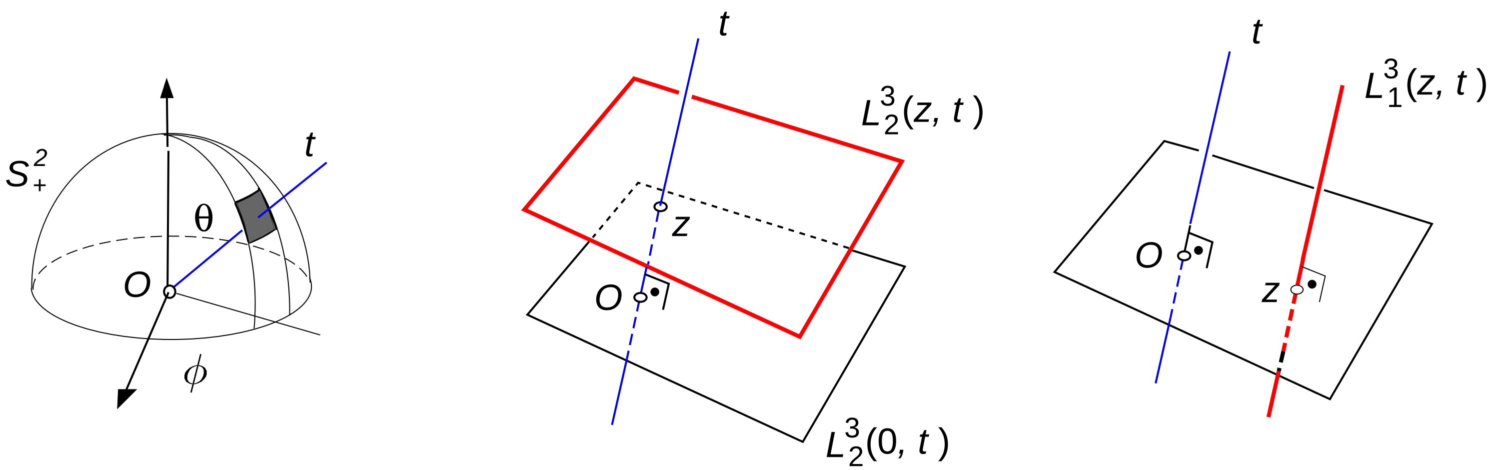
Recalling that $\mathbb{P}(\mathrm{d}z, \mathrm{d}t) = (\pi h)^{-1}\mathrm{d}z\,\mathrm{d}t$,

$$B = \frac{\pi}{2} h\, \mathbb{E} I(Y \cap \Lambda_{z,t}),$$

from which we can write an unbiased point estimator of the curve length B in terms of known constants and the observed intersection count,

$$\widehat{B} = \frac{\pi}{2} h\, I(Y \cap \Lambda_{z,t}).$$

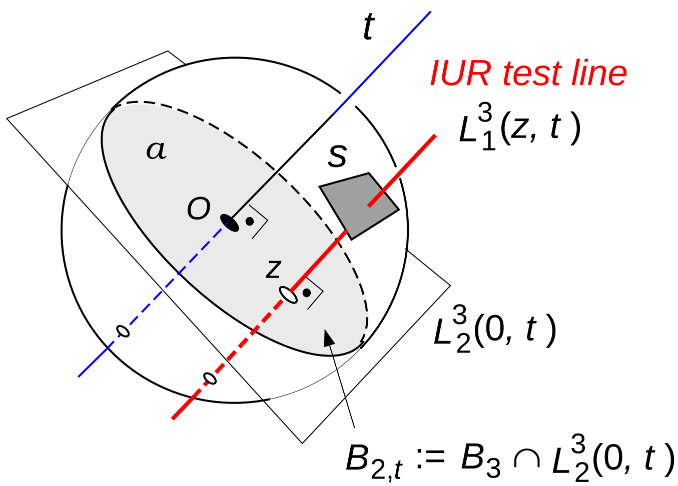
Invariant test lines and planes in $\mathbb{R}^3$

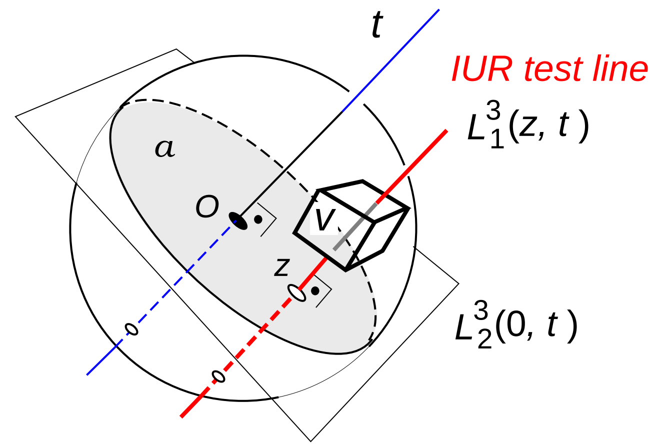


$$\mathrm{d}L_2^3 = \mathrm{d}z \wedge \mathrm{d}t, \quad z \in \mathbb{R}, \quad t \in \mathbb{S}_+^2, \quad \mathrm{d}t = \sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta, \quad \phi \in [0, 2\pi), \quad \theta \in [0, \pi/2],$$

$$\mathrm{d}L_1^3 = \mathrm{d}z \wedge \mathrm{d}t, \quad z \in \mathbb{R}^2, \quad t \in \mathbb{S}_+^2$$

Estimation of surface area and volume with test lines in $\mathbb{R}^3$. Preliminaries





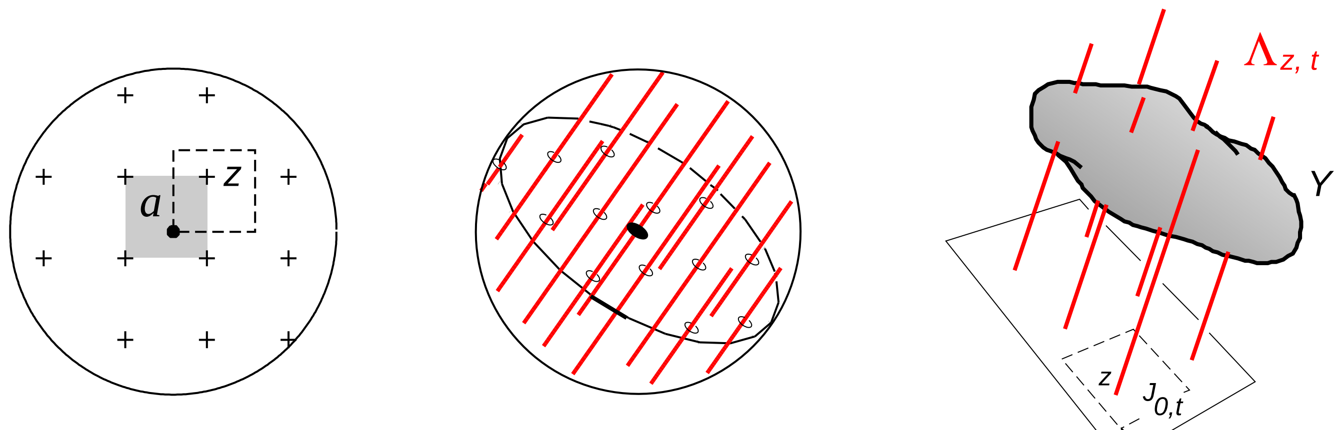
- $\partial y \in \mathbb{R}^3$: surface element of area s
- $y \in \mathbb{R}^3$: volume element of volume v,

$$\mathbb{P}(\mathrm{d}z, \mathrm{d}t) = \frac{\mathrm{d}z}{a} \cdot \frac{\mathrm{d}t}{2\pi}, \; z \in B_{2,t}, \; t \in \mathbb{S}_+^2,$$

$$\mathbb{E}I\Big(\partial y \cap L_1^3\Big) = \int_{L_1^3 \uparrow \partial y} I\Big(y \cap L_1^3(z,t)\Big)\, \mathbb{P}(\mathrm{d}z, \mathrm{d}t) = \frac{1}{2\pi a} \int_{\mathbb{S}_+^2} \mathrm{d}t \int_{\mathbb{R}^2} I\Big(y \cap L_1^3(z,t)\Big)\, \mathrm{d}z = \frac{1}{2\pi a} \int_{\mathbb{S}_+^2} s \cdot \cos\theta \, \mathrm{d}t = \frac{s}{2a},$$

$$\mathbb{E}L\Big(y \cap L_1^3\Big) = \int_{L_1^3 \uparrow y} L\Big(y \cap L_2^3(z,t)\Big)\, \mathbb{P}(\mathrm{d}z, \mathrm{d}t) = \frac{1}{2\pi a} \int_{\mathbb{S}_+^2} \mathrm{d}t \int_{\mathbb{R}^2} L\Big(y \cap L_2^3(z,t)\Big)\, \mathrm{d}z = \frac{1}{2\pi a} \int_{\mathbb{S}_+^2} v \, \mathrm{d}t = \frac{v}{a}.$$

Estimation of surface area and volume with the Fakir Probe



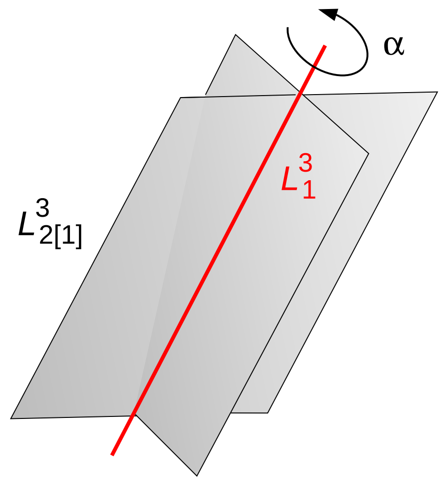
Fakir probe $\Lambda_{z,\,t}$

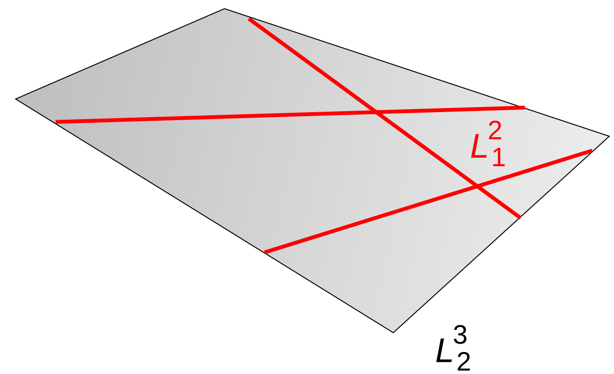
Making successive use of the *Monotone Convergence Theorem* and of *Santaló's Theorem*, the surface area and the volume of a bounded subset $Y \subset \mathbb{R}^3$ can be expressed as follows,

$$S(\partial Y) = 2a\,\mathbb{E}I(\partial Y \cap \Lambda_{z,t}), \qquad V(Y) = a\,\mathbb{E}L(Y \cap \Lambda_{z,t}),$$

from which the corresponding unbiased point estimators are straightforward.

An equivalence

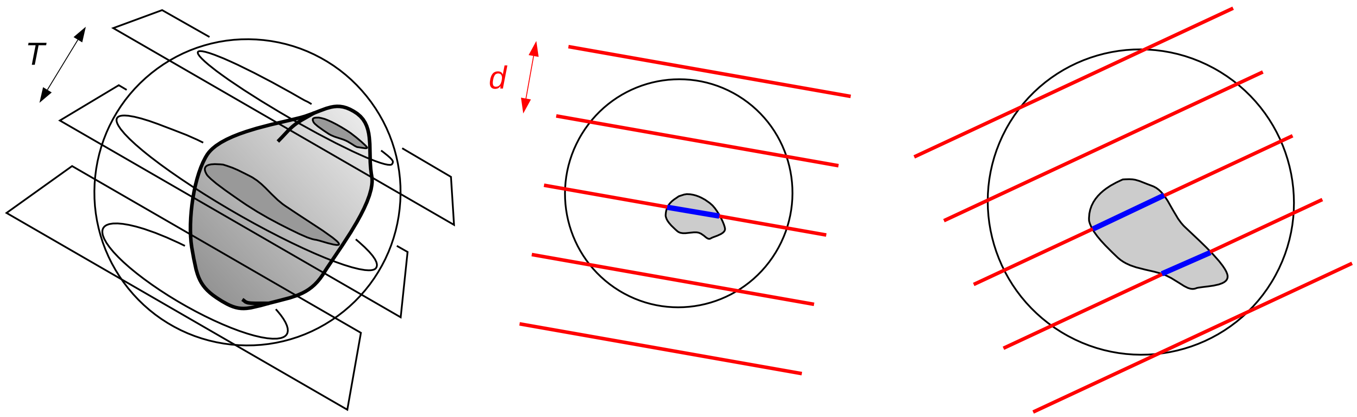




By virtue of the Petkantschin-Santaló identity, a motion invariant straight line in $\mathbb{R}^3$ is equivalent to a motion invariant straight line within a motion invariant plane:

$$\begin{aligned}
\mathrm{d}L_1^3(z,t) \wedge \mathrm{d}L_{2[1]}^3(\alpha) &= [\mathrm{d}z \wedge \mathrm{d}t] \wedge \mathrm{d}\alpha \\
&= [\mathrm{d}p \wedge \mathrm{d}q \wedge \mathrm{d}t] \wedge \mathrm{d}\alpha \\
&= [\mathrm{d}p \wedge \mathrm{d}t] \wedge [\mathrm{d}q \wedge \mathrm{d}\alpha] \\
&= \mathrm{d}L_2^3(p,t) \wedge \mathrm{d}L_1^2(q,\alpha).
\end{aligned}$$

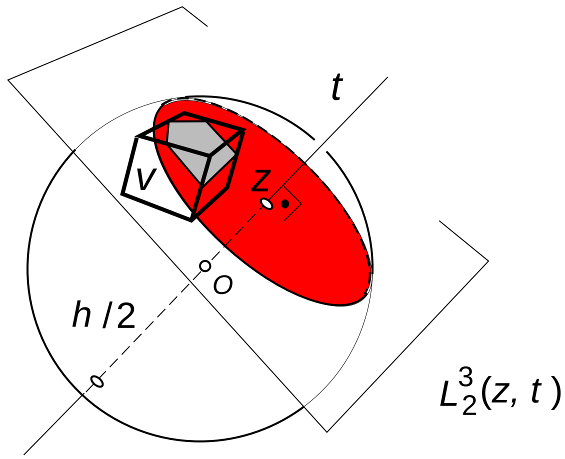
Application: IUR test lines on isotropic Cavalieri sections

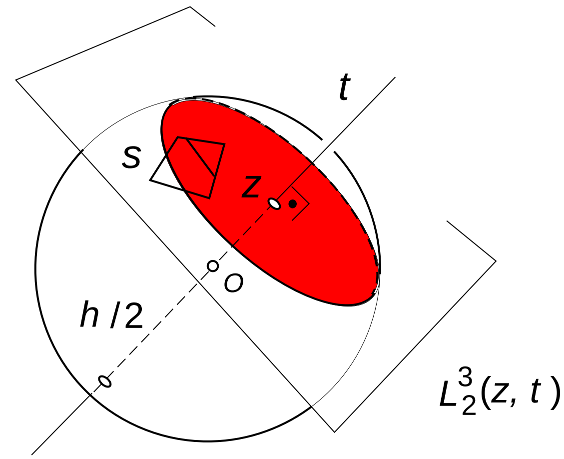


With the preceding results, application of *Santaló's Theorem* leads to the following representations,

$$S(\partial Y) = 2Td \cdot \mathbb{E}(I), \quad V(Y) = Td \cdot \mathbb{E}(L).$$

Estimation of volume and surface area with test planes in $\mathbb{R}^3$. Preliminaries



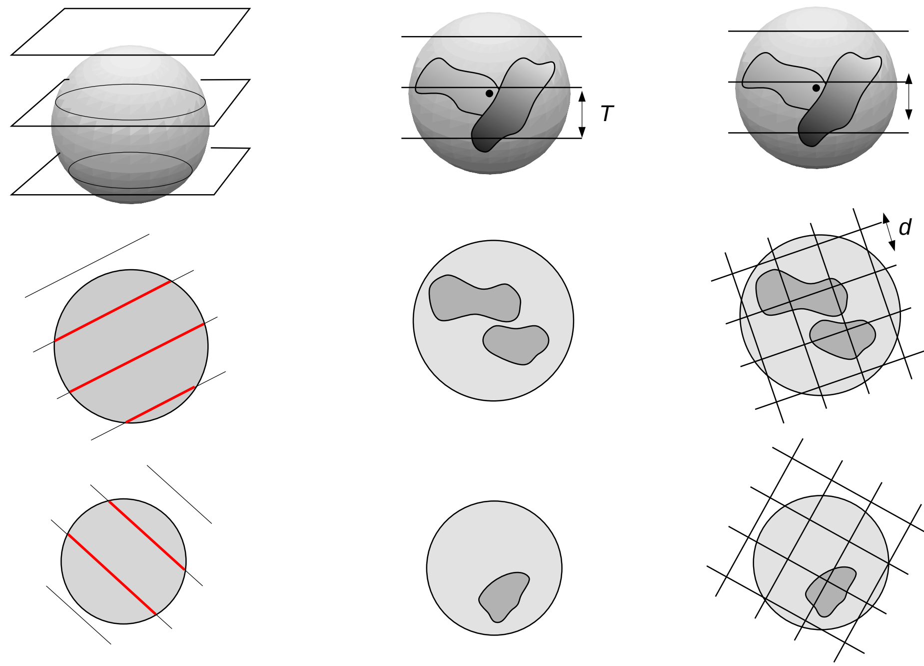


- $y \in \mathbb{R}^3$: volume element of volume v,
- $\partial y \in \mathbb{R}^3$: surface element of area s

$$\mathbb{E}A\Big(y \cap L_2^3\Big) = \int_{L_2^3 \uparrow y} A\Big(y \cap L_2^3(z,t)\Big)\,\mathbb{P}(\mathrm{d}z, \mathrm{d}t) = \frac{1}{2\pi h}\int_{\mathbb{S}_+^2} \mathrm{d}t \int_{\mathbb{R}} A\Big(y \cap L_2^3(z,t)\Big)\,\mathrm{d}z = \frac{1}{2\pi h}\int_{\mathbb{S}_+^2} v\,\mathrm{d}t = \frac{v}{h},$$

$$\mathbb{E}B\Big(\partial y \cap L_2^3\Big) = \int_{L_2^3 \uparrow \partial y} B\Big(\partial y \cap L_2^3(z,t)\Big)\,\mathbb{P}(\mathrm{d}z, \mathrm{d}t) = \frac{1}{2\pi h}\int_{\mathbb{S}_+^2} \mathrm{d}t \int_{\mathbb{R}} B\Big(\partial y \cap L_2^3(z,t)\Big)\,\mathrm{d}z = \frac{1}{2\pi h}\int_{\mathbb{S}_+^2} s \cdot \sin\theta\ \mathrm{d}t = \frac{\pi s}{4h}.$$

Estimation of surface area and volume with the isotropic Cavalieri design



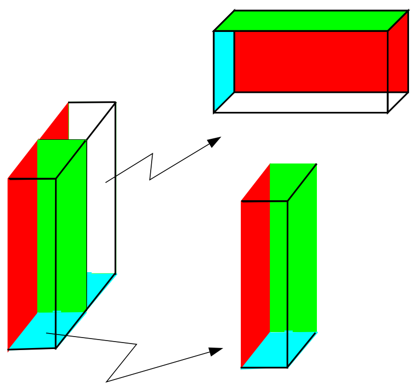
Making successive use of the *Monotone Convergence Theorem* and of *Santaló's Theorem*,

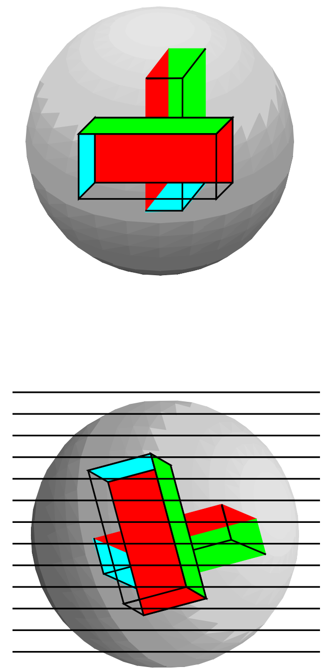
$$\widehat{S} = \frac{4}{\pi} \cdot T \cdot (B_1 + B_2 + \cdots + B_n) \text{ and } \widehat{V} = T \cdot (A_1 + A_2 + \cdots + A_n),$$

are unbiased estimators of the surface area and the volume of a bounded subset $Y \subset \mathbb{R}^3$, respectively. Further, if the sections are analysed with an IUR square grid test system of size d, then the corresponding two-stage unbiased estimators read,

$$\widetilde{S} = Td \cdot (I_1 + I_2 + \cdots + I_n) \text{ and } \widetilde{V} = Td^2 \cdot (P_1 + P_2 + \cdots + P_n).$$

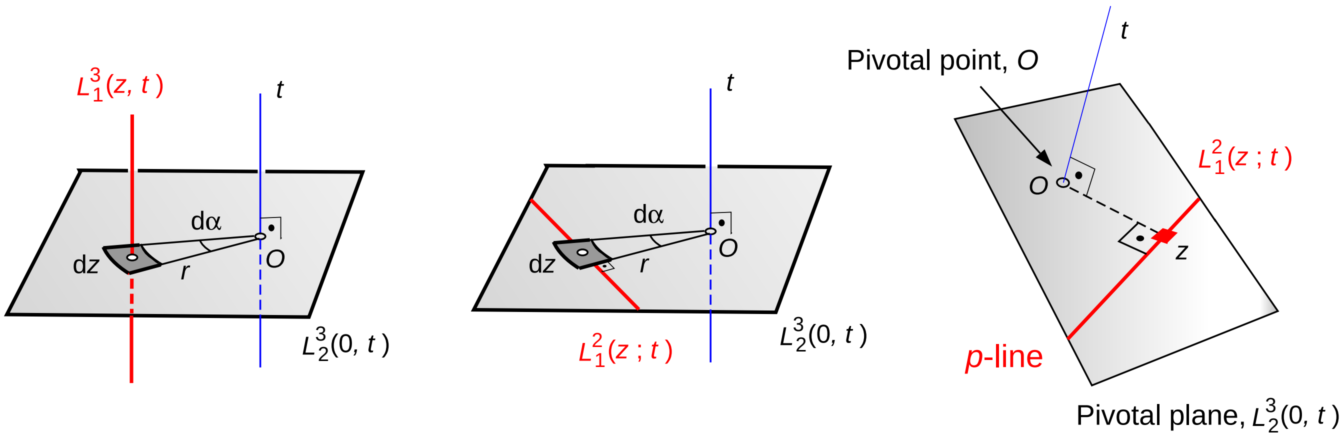
An aid to the isotropic Cavalieri design: The antithetic isector





The invariator principle and its applications

Invariator construction of a motion invariant test line in $\mathbb{R}^3$



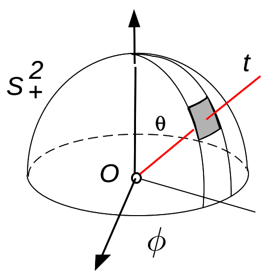
$$\mathrm{d}L_1^3 = \mathrm{d}z \wedge \mathrm{d}t$$

$$= r[\mathrm{d}r \wedge \mathrm{d}\alpha] \wedge \mathrm{d}t$$

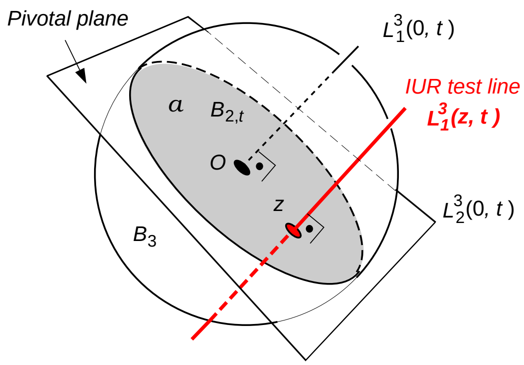
$$= r \cdot \mathrm{d}L_1^2 \wedge \mathrm{d}t$$

Thus, the invariator principle states that a point-weighted test line (= a *p*-line) $L_1^2(z;t)$ on a pivotal plane $L_2^3(0,t)$ is equivalent to a test line $L_1^3(z,t)$ with a motion invariant density in three dimensional space.

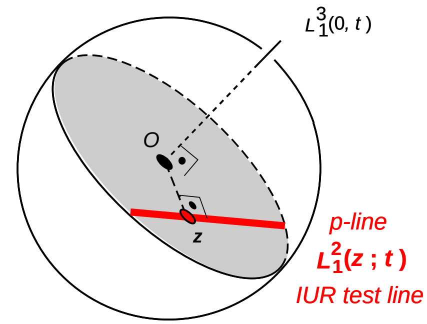
Probability element for a *p*-line



Isotropic orientation



Classical construction



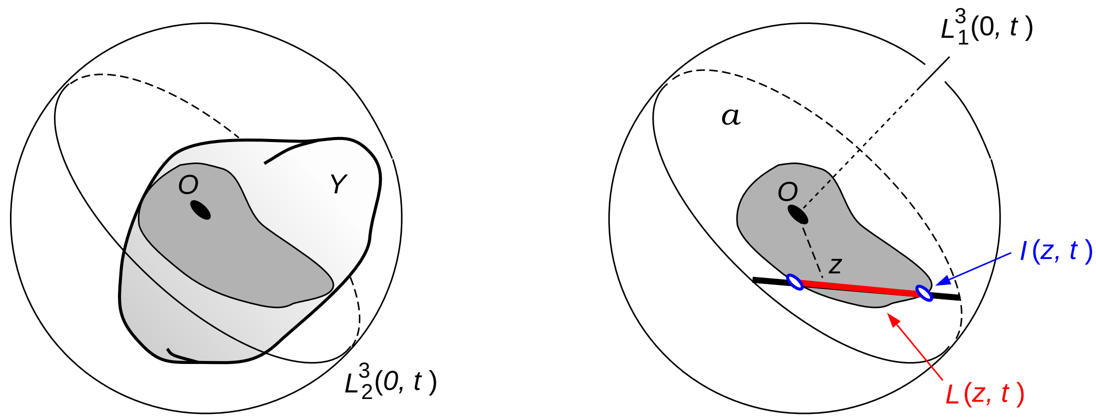
Invariator alternative

$$\mathrm{d}L_1^3(z,t) = \mathrm{d}z \wedge \mathrm{d}t, \quad \text{classical,}$$

$$\mathrm{d}L_1^3(z,t) = \mathrm{d}t \wedge \mathrm{d}L_1^2(z;t), \quad \text{invariator,}$$

$$\mathbb{P}(\mathrm{d}z, \mathrm{d}t) = \frac{\mathrm{d}z}{a} \cdot \frac{\mathrm{d}t}{2\pi}, \quad z \in B_{2,t},\ t \in \mathbb{S}_+^2$$

Surface area and volume with a *p*-line

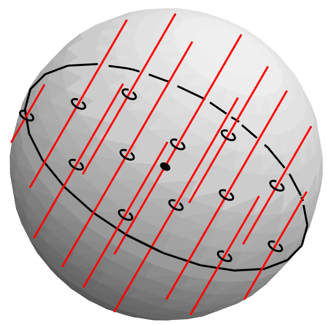


Crofton formulae:

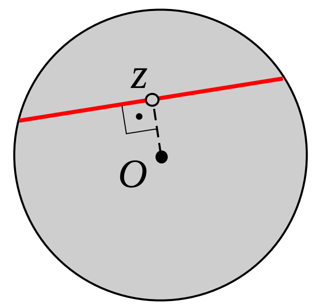
$$S(\partial Y) = \frac{1}{\pi}\int_{\mathbb{S}_+^2} \mathrm{d}t \int_{B_{2,t}} I(z,t)\,\mathrm{d}z = 2a\,\mathbb{E}I(z,t),\ \text{where } I(z,t) := \text{ number of intersections,}$$

$$V(Y) = \frac{1}{2\pi}\int_{\mathbb{S}_+^2} \mathrm{d}t \int_{B_{2,t}} L(z,t)\,\mathrm{d}z = a\,\mathbb{E}L(z,t),\ \text{where } L(z,t) := \text{ intercept length .}$$

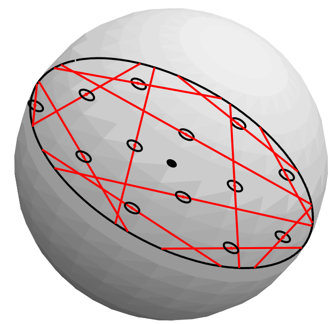
Surface area and volume with the Invariator Test Grid



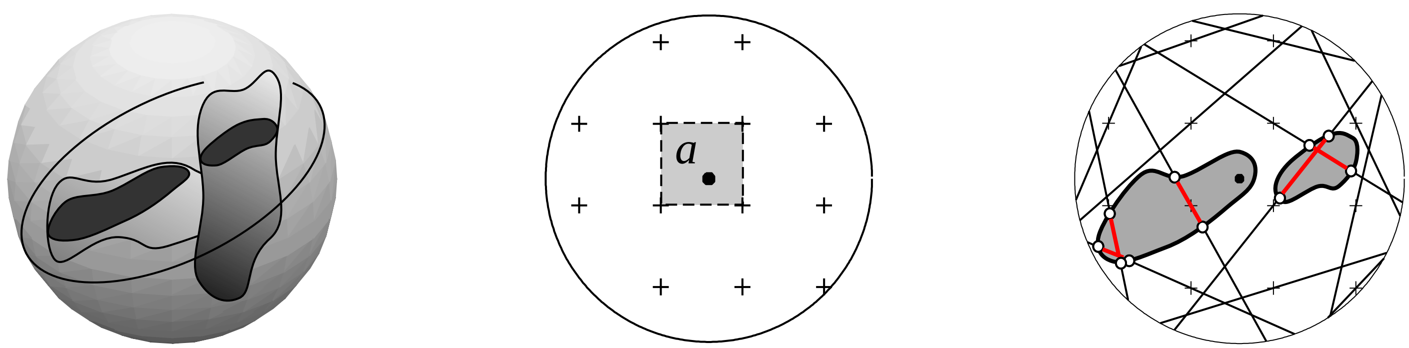
Fakir probe (classical motion invariant test lines)



p-line (invariator principle)



Invariator grid



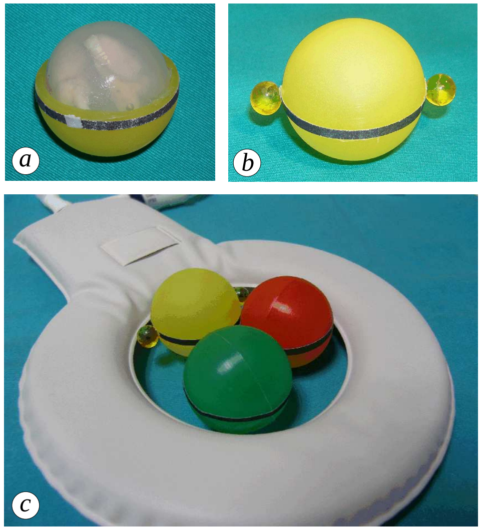
Implementation of the invariator grid

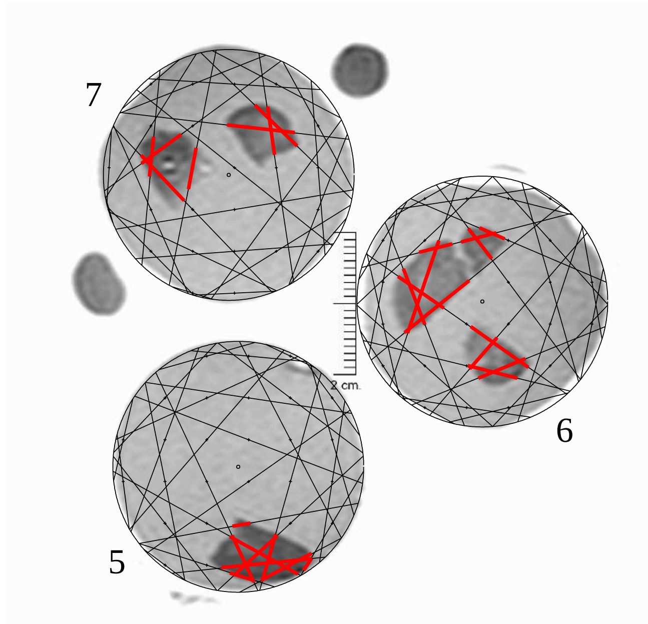
$$S(\partial Y) = 2a\,\mathbb{E}I(z,t), \qquad V(Y) = a\,\mathbb{E}L(z,t).$$

Application: rat brain surface area and volume with the Invariator

(Cruz-Orive LM, Ramos-Herrera ML & Artacho-Pérula E (2010) *J. Microscopy*)

Magnetic Resonance Imaging

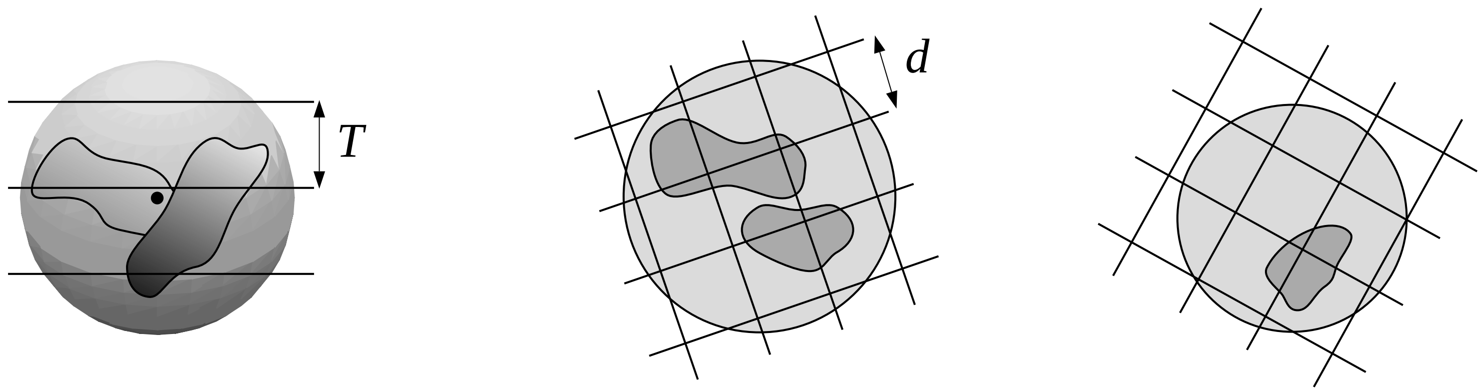




Unbiased estimators (with square grid generator of tile area a) for each brain:

$$\widehat{S} = 2aI, \qquad \widehat{V} = aL.$$

Classical isotropic Cavalieri design with antithetic arrangement



One stage UE's:

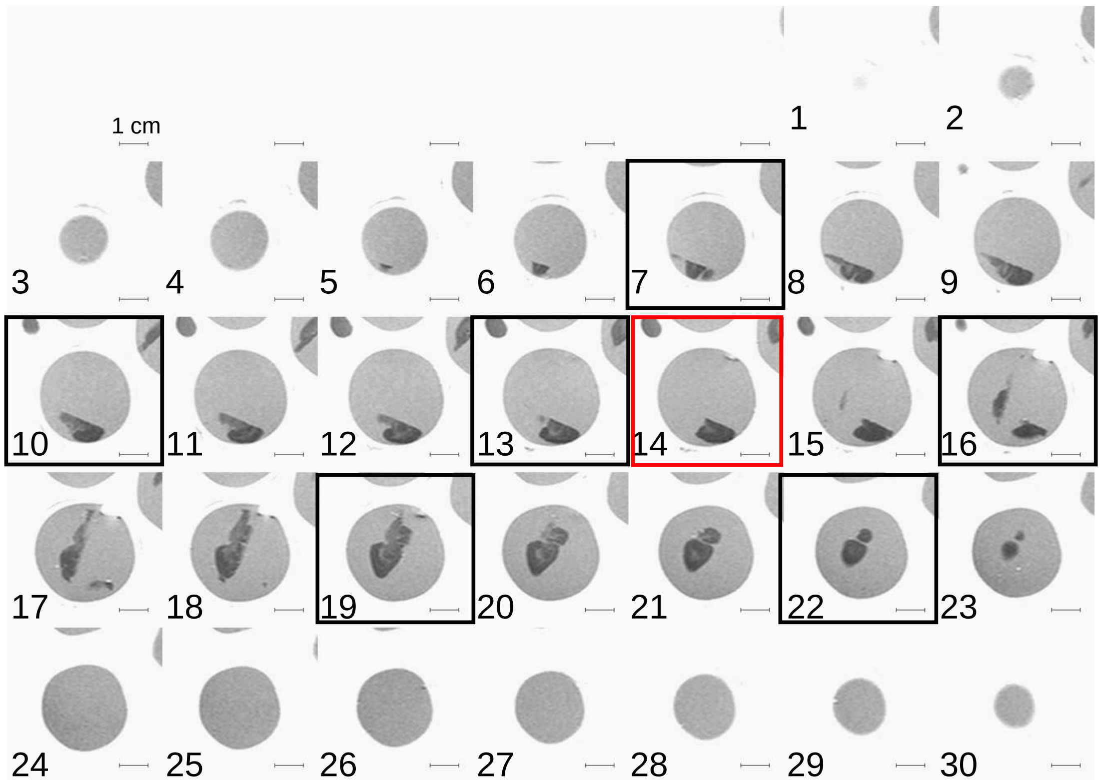
$$\widehat{S} = \frac{4}{\pi} T \sum B_i, \quad \widehat{V} = T \sum A_i \, .$$

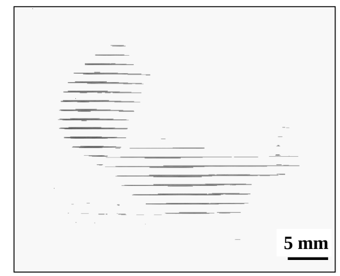
Two stage UE's (with square grid of size *d* to measure the sections):

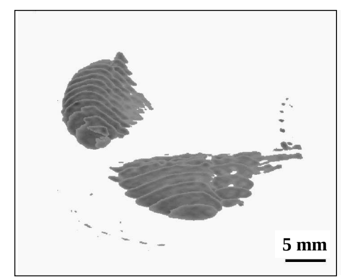
$$\widetilde{S} = \, Td \sum I_i, \quad \widetilde{V} = T \, d^2 \sum P_i.$$

Application to rat brain (contd.)

Exhaustive MRI isotropic Cavalieri series of brain #5 with antithetic arrangement of the two brain hemispheres. Slice thickness = 1.2 mm.

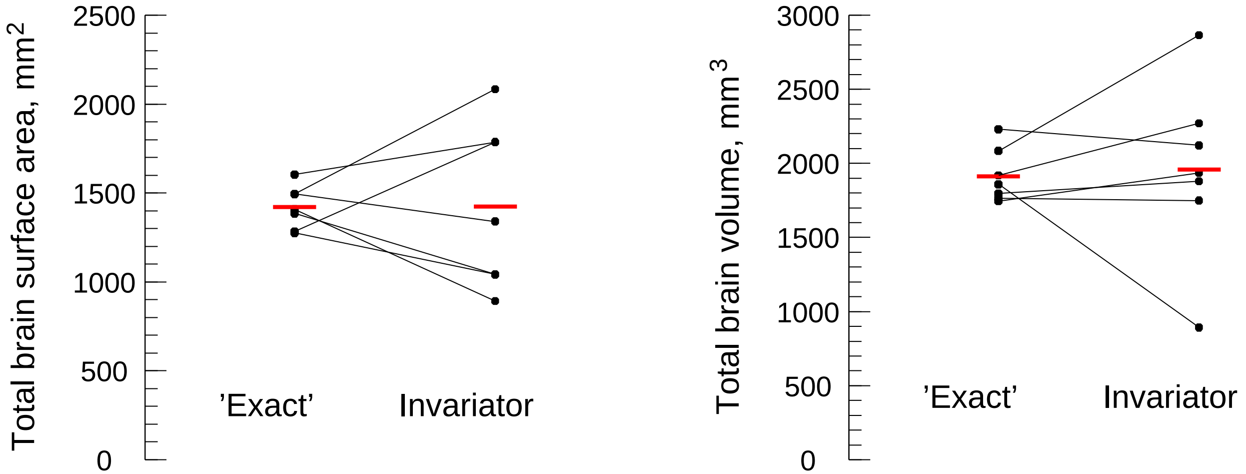






In this brain the approximately equatorial section #14 was used for the invariator. Sections {7, 10, 13, 16, 19, 22} constitute a systematic subsample of period T = 3.6 mm.

Rat brain surface area and volume: results

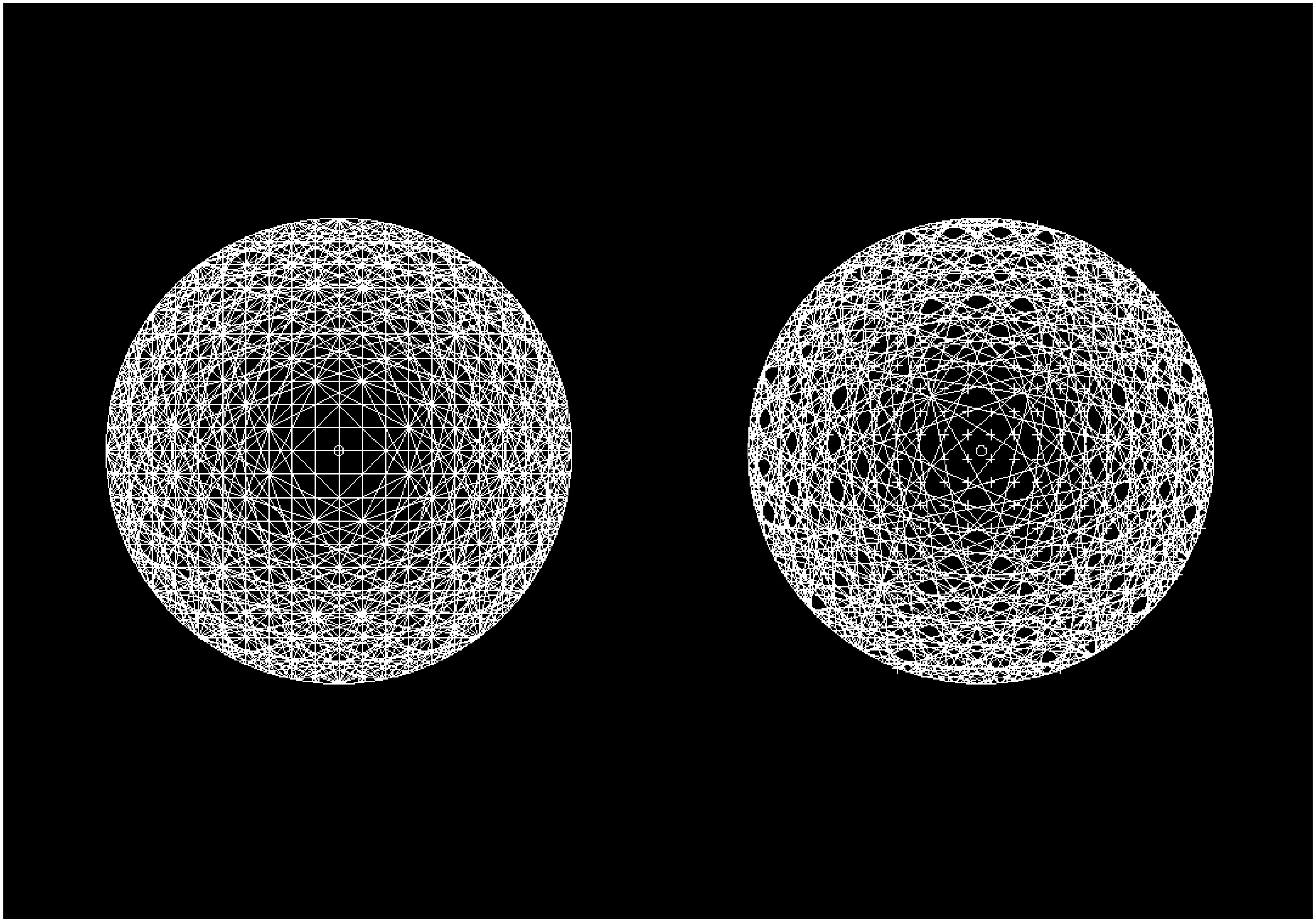


'Exact' := One stage estimators from isotropic Cavalieri series.

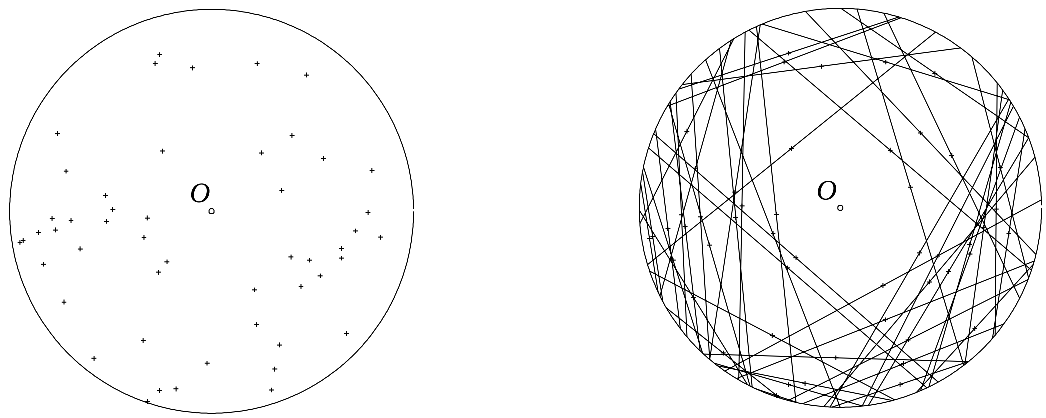
In this way the biological variance among brains could be estimated fairly accurately. Subtraction of the biological variance from the observed variance of the invariator data yielded a mean individual invariator $\mathrm{ce}(\cdot)$ of about 30% for both $\widehat{S}$ and $\widehat{V}$.

Conclusion: Use the invariator for cell populations, say, rather than individual organs.

The Pivotal Tessellation



The Poisson Pivotal Tessellation



For a connected region from the intersection of a Poisson pivotal tessellation with a disk of radius R we have,

$$\mathbb{E}\{N(R)\} = \text{ Mean number of vertices, or of sides} = 4 + O\Big(R^{-2}\Big),$$

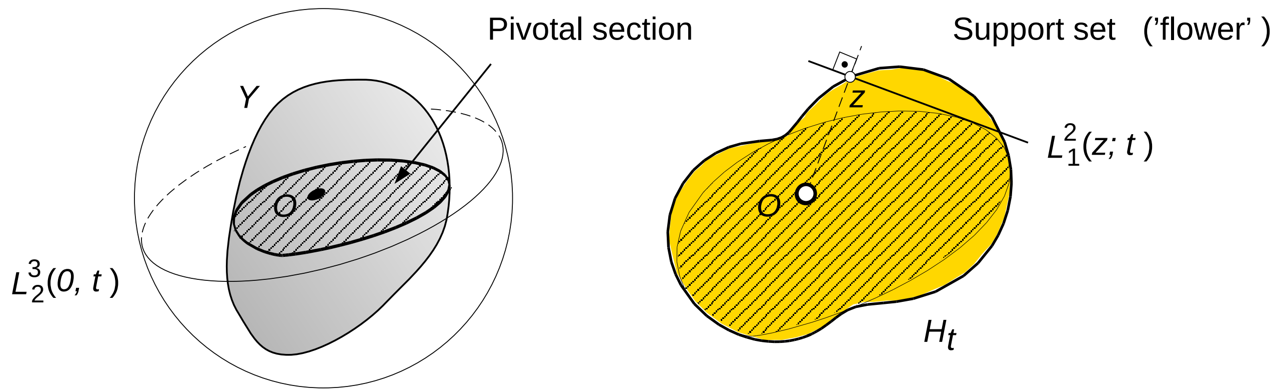
$$\mathbb{E}\{B(R)\} = \text{ Mean boundary length} = O\Big(R^{-1}\Big),$$

$$\mathbb{E}\{A(R)\} = \text{ Mean area} = O\Big(R^{-1}\Big).$$

INTERPRETATION ?

(Cruz-Orive (2009) *IAS* 28, 63–67.)

Case of a convex particle: Surface area from the 'FLOWER' of a pivotal section



$$S(\partial Y) = 2a\mathbb{E}I(z,t).$$

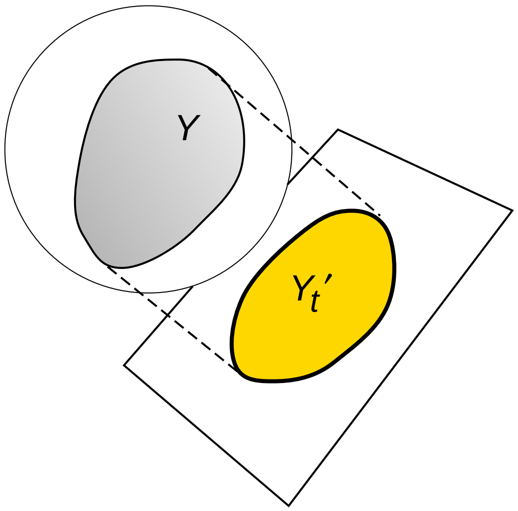
Flower of the convex pivotal section $Y \cap L_1^2(z;t)$: $\quad H_t := \left\{z \in \mathbb{R}^2 : Y \cap L_1^2(z;t) \neq \emptyset\right\}$.

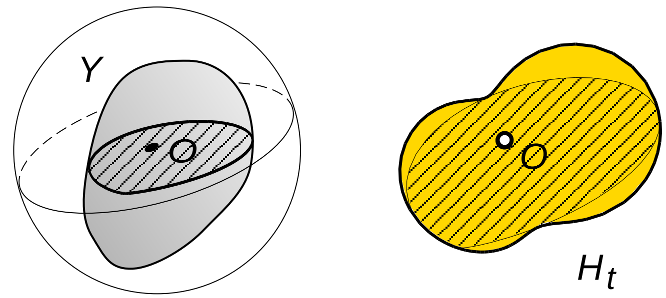
$$\therefore I(z,t) = \begin{cases} 2 & \text{with probability } A(H_t)/a, \\ 0 & \text{with probability } 1 - A(H_t)/a, \end{cases}$$

$$\therefore S(\partial Y) = 4\,\mathbb{E}A(H_t).$$

Interlude: A duality

Y convex





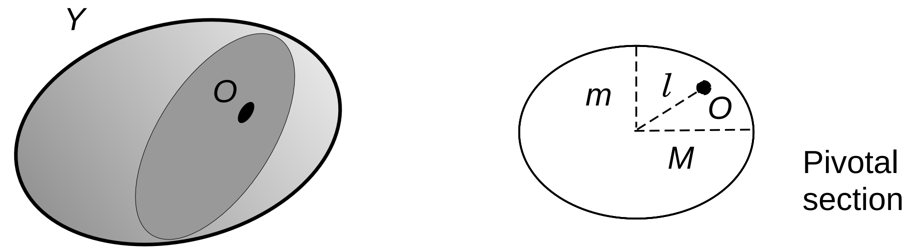
$$S(\partial Y) = 4\,\mathbb{E}A(Y_t')$$

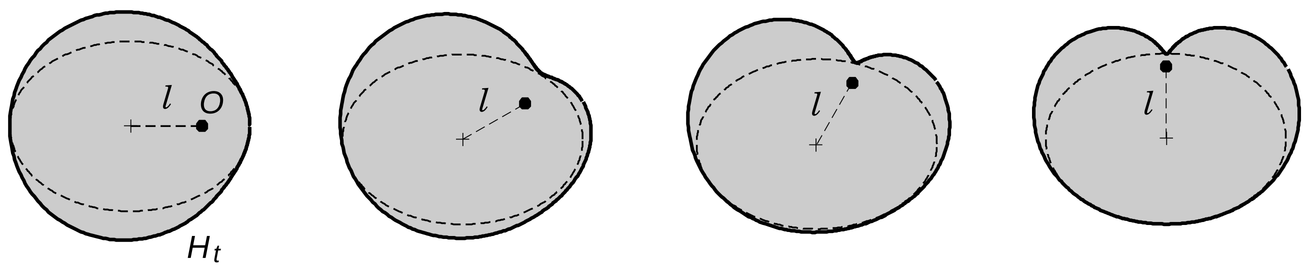
Cauchy projection formula

$$S(\partial Y) = 4\,\mathbb{E}A(H_t)$$

Flower formula

Special case: Surface area of a triaxial ellipsoid





Flowers of identical areas if l is fixed

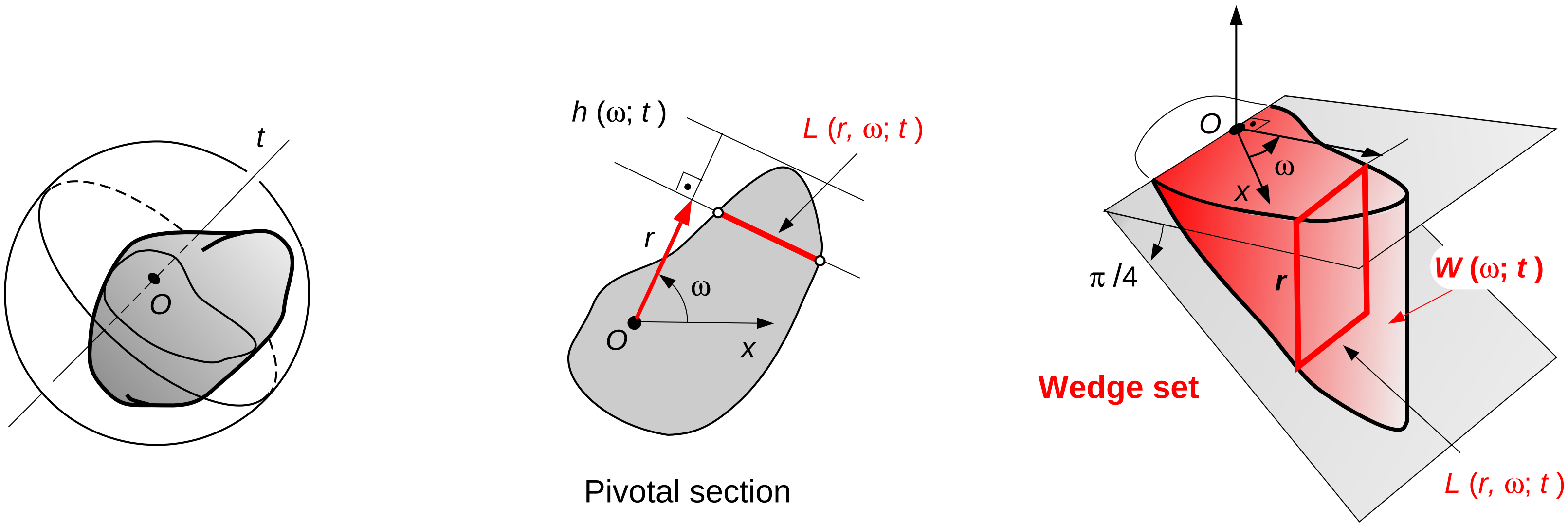
$$S(\partial Y) = 4\,\mathbb{E}\{A(H_t)\}, \quad A(H_t) = \frac{\pi}{2}\big(M^2 + m^2 + l^2\big).$$

$$\therefore S(\partial Y) = 2\pi\,\mathbb{E}\big(M^2 + m^2 + l^2\big).$$

Remark:

$$A(H_K) = \frac{1}{2}\int_0^{2\pi} h_K^2(\omega)\,\mathrm{d}\omega, \;\; O \in K \subset \mathbb{R}^2, \text{ convex.}$$

Volume of an arbitrary particle via the mean WEDGE volume



Pivotal section

$$V(Y) = 2\pi\,\mathbb{E}V(W_t).$$

Hint of proof. Use Crofton's formula with $\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\omega$,

$$V(Y) = \frac{1}{2\pi}\int_{\mathbb{S}^2_+}\mathrm{d}t\int_0^{2\pi}\mathrm{d}\omega\int_0^{h(\omega;t)} L(r,\omega;t)\,r\mathrm{d}r.$$

But the integral

$$\int_0^{h(\omega;t)} L(r,\omega;t)\,r\mathrm{d}r$$

can be interpreted as the volume of a well defined 45° ' WEDGE SET' $W(\omega;t)$. The result follows if, or each pivotal section, i.e. for each $t \in \mathbb{S}^2_+$, we set $V(W_t) := \mathbb{E}V\{W(\omega;t)\}$.

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