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Volumes of convex hulls of $n \leq d+1$ points in d-dimensional convex bodies

The convex hull of $n \leq d+1$ points randomly chosen from a d-dimensional convex body K according to the uniform distribution in K forms an $(n-1)$-dimensional simplex with probability 1. We denote its volume by $V_{K[n]}$ and ask if for two d-dimensional convex bodies K and L and $k > 0$, $K \subseteq L$ implies $\mathbb{E}V_{K[n]}^k \leq \mathbb{E}V_{L[n]}^k$.

In 2006, M. Meckes raised the question whether $K \subseteq L$ would imply $\mathbb{E}V_{K[d+1]} \leq \mathbb{E}V_{L[d+1]}$. L. Rademacher gave an answer for the case $n = d+1$ and $k = 1$, when $d \neq 3$. Higher moments were investigated by Rademacher and Reichenwallner & Reitzner, only leaving the expected volume of a random tetrahedron in dimension three as an open task.

We give a similar result for $n < d+1$, showing that for $k > 0$ there exist two d-dimensional convex bodies K and L satisfying $K \subseteq L$, but $\mathbb{E}V_{K[n]}^k > \mathbb{E}V_{L[n]}^k$, unless $n \in \{3, 4\}$ and $k = 1$.