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Second-order pseudo-stationary random fields and point processes on graphs and their edges

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Consider an undirected connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with countably many vertices and edges so that to each edge there is a bijective mapping from the edge to a finite non-empty open interval. This allows us to treat the edges as one-dimensional line segments. Assuming that $\mathcal{V}$ and any edge $e \in \mathcal{E}$ are disjoint and that the edges are pairwise disjoint, we call the whole graph set $L = \mathcal{V} \cup \bigcup_{e \in \mathcal{E}} e$ a linear network. Linear networks have been considered in connection to for example road networks, dendrite networks of neurons, and brick walls. The edge coordinates lead naturally to a geodesic metric $d_{\mathcal{G}}(u, v)$ on L given by shortest path distance.

Our main goal is to establish sufficient conditions on the existence of positive definite functions of the form $K(d_{\mathcal{G}}(u, v))$ for all $u, v \in L$. Then the Kolmogorov Extension Theorem establishes the existence of a separable (Gaussian) random field $Z = \{Z(u) : u \in S\}$ with covariance function

$$\operatorname{cov}(Z(u), Z(v)) = K\left(d_{\mathcal{G}}(u, v)\right) \quad \forall u, v \in L.$$

We say then that the covariance function is **pseudo-stationary** and that the random field Z is **second-order pseudo-stationary**, noticing that we do not require that the mean function $\mathrm{E}Z(u)$ is constant. Note that our setting is different from that in research on random fields on directed trees such as in a network of rivers or streams where water flows in one direction. Then special techniques are appropriate for constructing covariance functions of the form above. However, our techniques will be different, since we deal with undirected graphs.

One motivation for considering a second-order pseudo-stationary random field Z is that for any geodesic path $p_{uv} \subseteq L$ connecting two points $u, v \in L$, the restriction of Z to p_{uv} has the same covariance structure as the random field $\tilde{Z}(t)$ defined on $t \in [0, t_0] \subset \mathbb{R}$ where $\operatorname{cov}(\tilde{Z}(t), \tilde{Z}(s)) = K(|t - s|)$ and $t_0 = d_{\mathcal{G}}(t, s)$. In brief, Z restricted to a geodesic path is indistinguishable from a corresponding Gaussian random field on a closed interval.

Another motivation is that given a covariance function of the form above, we can construct second-order pseudo-stationary point processes on $\mathcal{G}$, meaning that the point process has a pair correlation function of the form $g(u, v) = g_0(d_{\mathcal{G}}(u, v))$ for all $u, v \in L$. A Poisson process on L is second-order pseudo-stationary point processes but to the best of our knowledge, apart from the Poisson process, models for second-order pseudo-stationary point processes on linear networks have not been specified in the literature. We show that for a log Gaussian Cox process X, i.e. when X conditional on a Gaussian random field Z on L is a Poisson process with intensity function $\exp(Z(u))$, $u \in L$, second-order pseudo-stationarity of Z is equivalent to pseudo-stationarity of X. Further examples of second-order pseudo-stationary point processes on linear networks will be discussed in the talk.